

MINIMAL MODELS FOR KÄHLER THREEFOLDS

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ABSTRACT. Let X be a compact Kähler threefold that is not uniruled. We prove that X has a minimal model.

1. INTRODUCTION

The minimal model program (MMP) is one of the cornerstones in the classification theory of complex projective varieties. It is fully developed only in dimension 3, despite tremendous recent progress in higher dimensions, in particular by [BCHM10]. In the analytic Kähler situation the basic methods from the MMP, such as the base point free theorem, fail. Nevertheless it seems reasonable to expect that the main results should be true also in this more general context.

The goal of this paper is to establish the minimal model program for Kähler threefolds X whose canonical bundle K_X is pseudo-effective. To be more specific, we prove the following result:

1.1. Theorem. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities. Suppose that K_X is pseudo-effective or, equivalently, that X is not uniruled. Then X has a minimal model, i.e. there exists a MMP*

$$X \dashrightarrow X'$$

such that $K_{X'}$ is nef.

If X is a projective threefold, this statement was established by the seminal work of Kawamata, Kollár, Mori, Reid and Shokurov [Mor79, Mor82, Rei83, Kaw84a, Kaw84b, Kol84, Sho85, Mor88, KM92]. Based on the deformation theory of rational curves on smooth threefolds [Kol91b, Kol96], Campana and the second-named author started to investigate the existence of Mori contractions [CP97, Pet98, Pet01]. Nakayama [Nak02] proved the existence of an (even good) minimal model for a Kähler threefold of algebraic dimension 2.

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The equivalence of X being non-uniruled and K_X being pseudo-effective for a threefold X is a remarkable result due to Brunella [Bru06]¹. This settles in particular the problem of the existence K_X -negative rational curves in case K_X is not pseudo-effective. If K_X is pseudo-effective, but not nef, ad hoc methods provide K_X -negative rational curves, as we will show in this paper. One main step in the MMP is the construction of extremal contractions, which in the projective case follows from the base-point free theorem. In the Kähler case we need a completely different approach.

Restricting from now on to varieties X with K_X pseudo-effective, we consider the divisorial Zariski decomposition [Bou04]

$$K_X = \sum_{j=1}^r \lambda_j S_j + N(K_X).$$

Here $N(K_X)$ is an \mathbb{R} -divisor which is “nef in codimension one”. If $K_X|_{S_j}$ is not pseudoeffective we use a (sub-)adjunction argument to show that S_j is uniruled. We can then prove that K_X is not nef (in the sense of [DPS94]) if and only if there exists a curve $C \subset X$ such that $K_X \cdot C < 0$. In Section 5 we show how deformation theory on the threefold X and the (maybe singular) surfaces S_j can be used to establish an analogue of Mori’s bend and break technique. As a consequence we derive the cone theorem for the Mori cone $\overline{\text{NE}}(X)$ (cf. Theorem 6.2).

However the Mori cone $\overline{\text{NE}}(X)$ is not the right object to consider: even if we find a bimeromorphic morphism $X \rightarrow Y$ contracting exactly the curves lying on some K_X -negative extremal ray in $\overline{\text{NE}}(X)$, it is not clear that Y is a Kähler space. However it had been observed in [Pet98] that the Kähler could be preserved if we contract extremal rays in $\overline{\text{NA}}(X)$, the cone generated by positive closed currents of bidimension $(1, 1)$. Based on the description of $\overline{\text{NA}}(X)$ by Demailly and Păun [DP04], we prove the following cone theorem:

1.2. Theorem. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a countable family $(\Gamma_i)_{i \in I}$ of rational curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq 4$$

and

$$\overline{\text{NA}}(X) = \overline{\text{NA}}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$$

We then proceed to prove the existence of contractions of an extremal ray $\mathbb{R}^+[\Gamma_i]$. If the curves $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$ cover a divisor S , we can use a generalisation of Grauert’s criterion [Gra62] by Ancona and Van Tan [AT84]. If the curves in the extremal ray cover only a 1-dimensional set C (i.e. the contraction, if it exists, is small), the problem is more subtle. By Grauert’s criterion it is sufficient and necessary to find an ideal sheaf

¹If X is projective this is known in arbitrary dimension by [BDPP04].

$\mathcal{I} \subset \mathcal{O}_X$ such that the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is ample and has support on C . In practice it is very difficult to compute the conormal sheaf, even for the reduced curve C . However since the curves in C belong to an extremal ray there exists a nef and big cohomology class α which is zero exactly on the curves in $\mathbb{R}^+[\Gamma_i]$; the class α is the analogon of the nef supporting divisor in the projective case. Considering once again the divisorial Zariski decomposition $K_X = \sum_{j=1}^r \lambda_j S_j + N(K_X)$ we now make a case distinction. If there exists a surface S_j such that $S_j \cdot C < 0$ this gives one direction where the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is ample. Moreover we prove that $\alpha|_{S_j}$ is nef and big, so an application of the Hodge index theorem yields another direction where $\mathcal{I}/\mathcal{I}^2$ is ample.

Thus we are left with the case where $N(K_X) \cdot C < 0$. If X is projective, Nakayama [Nak04, III, 4.b] gives a very short argument: if H is an ample divisor, some multiple of the class $N(K_X) + \varepsilon H$ with $0 < \varepsilon \ll 1$ gives a linear system without fixed component, so C is contained in a lci curve having ample conormal bundle along C , so we conclude as in the first case. In the non-algebraic case we use again the deep results by Demailly-Păun [DP04] and Boucksom [Bou04] to prove that there exists a modification $\mu : \tilde{X} \rightarrow X$ and a Kähler form $\tilde{\alpha}$ such that $\mu_*\tilde{\alpha} = \alpha$. Analysing the positivity of the μ -exceptional divisor we construct an ideal sheaf \mathcal{I} having the required properties. In summary we have proven the contraction theorem:

1.3. Theorem. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X -negative extremal ray in $\overline{NA}(X)$. Then the contraction of $\mathbb{R}^+[\Gamma_i]$ exists in the Kähler category (cf. Definition 3.17).*

Since Mori's theorem [Mor88] assures the existence of flips also in the analytic category, we can now run the MMP and obtain Theorem 1.1.

By [DP03, Thm.0.3] this also implies that the non-vanishing conjecture holds for compact Kähler threefolds:

1.4. Corollary. *Let X be a normal \mathbb{Q} -factorial compact (non-projective) Kähler threefold with at most terminal singularities. Then X is uniruled if and only if $\kappa(X) = -\infty$.*

Actually we obtain a little more, using [Pet01]:

1.5. Corollary. *Let X be a normal \mathbb{Q} -factorial compact (non-projective) Kähler threefold with at most terminal singularities. Suppose that K_X is nef. Then mK_X is spanned for some positive m , unless (possibly) there is no positive-dimensional subvariety through the very general point of X and X is not bimeromorphic to T/G where T is a torus and G a finite group acting on T .*

The remaining problem to solve abundance for Kähler threefolds completely is to prove the following well-known

1.6. Conjecture. *Let X be a smooth compact Kähler threefold or a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities. Assume there is no positive-dimensional subvariety through the very general point of X . Then X is bimeromorphic to T/G with T a torus and G a finite group acting on T .*

We hope to come back on this problem in a sequel to this paper.

We end the introduction with a discussion on uniruled non-algebraic Kähler threefolds. Here the global bimeromorphic structure is rather simple; as usual, $a(X)$ denotes the algebraic dimension. Since X is uniruled and non-algebraic, the rational quotient (MRC-fibration) is an almost holomorphic map $f : X \dashrightarrow S$ over a smooth non-algebraic surface S .

1.7. Theorem. [CP00] *Let X be a smooth compact Kähler threefold that is uniruled and not projective. Then X falls in one of the following classes.*

- a) $a(X) = 0$ and there is an almost holomorphic \mathbb{P}_1 -fibration $f : X \dashrightarrow S$ over a smooth surface S with $a(S) = 0$.
- b) $a(X) = 1$ and we are in one of the following subcases.
 - (i) X is bimeromorphic to $\mathbb{P}_1 \times F$ with $a(F) = 0$.
 - (ii) the algebraic reduction $X \rightarrow B$ is holomorphic onto a smooth curve B and the general fibre is of the form $\mathbb{P}(\mathcal{O} \oplus L)$ with L a topologically trivial line bundle over an elliptic curve which is not torsion.
- c) $a(X) = 2$, X has a meromorphic elliptic fibration over an algebraic surface.

We expect that the MMP for uniruled threefolds would exhibit a birational X' which is a Mori fibre space $\varphi : X' \rightarrow S$ over a non-algebraic surface S .

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2. BASIC NOTIONS

We will use frequently standard terminology of the minimal model program (MMP) as explained in [KM98] or [Deb01]. In particular we say that a normal complex space X is \mathbb{Q} -factorial if for every Weil divisor D there exists an integer $m \in \mathbb{N}$ such that $\mathcal{O}_X(mD)$ is a locally free sheaf, i.e. mD is a Cartier divisor.

Somewhat abusively we will denote the cycle space or Barlet space of a complex space X by $\text{Chow}(X)$. A curve C is a one-dimensional projective complex space that is irreducible and reduced. If X is a complex space, we

²ANR-10-JCJC-0111

will denote an effective 1-cycle by $\sum \alpha_l C_l$ where the $C_l \subset X$ are curves and the coefficients α_l are positive integers, unless explicitly mentioned otherwise.

2.1. Definition. *Let X be an irreducible and reduced complex space. A differential form ω of type (p, q) is a differential form (of class C^∞) of type (p, q) on the smooth locus X_{nons} such that for every point $x \in X_{\text{sing}}$ there exists an open neighbourhood $x \in U \subset X$ and a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$ such that there exists a differential form ω_V of type (p, q) and class C^∞ on V such that $\omega_V|_{U \cap X_{\text{nons}}} = \omega|_{U \cap X_{\text{nons}}}$.*

We denote the sheaf of (p, q) -forms by $\mathcal{A}_X^{p, q}$, the subsheaf of forms with compact support is denoted by $\mathcal{A}_c^{p, q}$. We define analogously forms of degree r and denote the corresponding sheaf by \mathcal{A}_X^r . We denote by $\mathcal{A}_{\mathbb{R}, X}^r$ the sheaf of real forms of degree r . For details, more information and references we refer to [AG06, p.181] and [Dem85].

Let X be an irreducible and reduced complex space of dimension n . Then the notion of a current of degree s (resp. bidegree (p, p) or bidimension $(n-p, n-p)$) is well-defined, see [Dem85], as well as the notion of closedness and positivity. We will denote by $\mathcal{D}^{p, q}(X)$ the space of currents of bidegree (p, q) .

Let now X be a normal complex space and ω a form of type (p, q) on X . Let $\pi : \hat{X} \rightarrow X$ be a morphism from a complex manifold \hat{X} . The pull-back $\pi^*\omega$ is then defined as follows: every point $x \in X$ has an open neighborhood U admitting a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$ and there exists a form ω_V on V such that $\omega_V|_{U \cap X_{\text{nons}}} = \omega|_{U \cap X_{\text{nons}}}$. We can thus define the pull-back $\pi^*\omega$ on $\pi^{-1}(U)$ by setting

$$(\pi^*\omega)|_{\pi^{-1}(U)} := (i_U \circ \pi|_{\pi^{-1}(U)})^* \omega_V.$$

By [Dem85, Lem.1.3.] these local definitions glue together, so we obtain a pull-back map:

$$\pi^* : \mathcal{A}^{p, q}(X) \rightarrow \mathcal{A}^{p, q}(\hat{X}).$$

In particular if π is proper, we can define the direct image map

$$\pi_* : \mathcal{D}^{p, q}(\hat{X}) \rightarrow \mathcal{D}^{p, q}(X), \quad T \mapsto \pi_*(T)$$

by setting

$$\pi_*(T)(\omega) = T(\pi^*(\omega)) \quad \forall \omega \in \mathcal{A}_c^{n-p, n-q}(X).$$

The notion of a singular Kähler space was first introduced by Grauert [Gra62].

2.2. Definition. *An irreducible and reduced complex space X is Kähler if there exists a Kähler form ω , i.e. a positive closed real $(1, 1)$ -form $\omega \in \mathcal{A}_{\mathbb{R}}^{1, 1}(X)$ such that the following holds: for every point $x \in X_{\text{sing}}$ there exists an open neighbourhood $x \in U \subset X$ and a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$, and a strictly plurisubharmonic C^∞ -function $f : V \rightarrow \mathbb{C}$ with $\omega|_{U \cap X_{\text{nons}}} = (i\partial\bar{\partial}f)|_{U \cap X_{\text{nons}}}$.*

2.3. Remark. [Var89, 1.3.1] If X is a compact Kähler space and $\pi : \hat{X} \rightarrow X$ a projective morphism, then \hat{X} is again Kähler. In particular, X has a Kähler desingularisation. A subvariety of a Kähler space is also Kähler.

3. THE DUAL KÄHLER CONE

Since we will work on complex spaces which a priori might not be Kähler, we will use $\partial\bar{\partial}$ -cohomology.

3.1. Definition. Let X be a normal complex space. We set

$$H_{\partial\bar{\partial}}^{1,1}(X) = \{\omega \in \mathcal{A}^{1,1}(X) \mid d\omega = 0\} / \partial\bar{\partial}\mathcal{A}^0(X).$$

Given a form $\omega \in \mathcal{A}^{1,1}(X)$, its $\partial\bar{\partial}$ -class is denoted by $\{\omega\}$.

3.2. Remark. We can also define the $\partial\bar{\partial}$ -class of a d -closed $(1,1)$ -current α by saying that α is equivalent to α' if we have

$$\alpha = \alpha' + \partial\bar{\partial}f$$

with f a current of degree 0. If X is smooth, it is well-known [GH78, p.328ff] that this definition yields the same $\partial\bar{\partial}$ -cohomology group.

Note that if $\pi : \hat{X} \rightarrow X$ is a desingularisation, we have a well-defined pull-back

$$\pi^* : H_{\partial\bar{\partial}}^{1,1}(X) \rightarrow H_{\partial\bar{\partial}}^{1,1}(\hat{X}), \quad \{\omega\} \mapsto \{\pi^*\omega\}.$$

Indeed the pull-back of forms commutes with the differential operators d and $\partial\bar{\partial}$ [Dem85, Sect.1].

3.3. Lemma. Let X be a normal complex space, and let $\pi : \hat{X} \rightarrow X$ be a desingularisation. Then the pull-back

$$\pi^* : H_{\partial\bar{\partial}}^{1,1}(X) \rightarrow H_{\partial\bar{\partial}}^{1,1}(\hat{X})$$

is injective.

Proof. Let ω be a d -closed $(1,1)$ -form on X such that $\{\pi^*\omega\}$ is zero in $H_{\partial\bar{\partial}}^{1,1}(\hat{X})$. Then we have

$$\pi^*\omega = \partial\bar{\partial}\hat{f},$$

with $\hat{f} \in \mathcal{A}^0(\hat{X})$. Let now F be any irreducible component of a π -fibre, taken with the reduced structure. Let $i : F \rightarrow \hat{X}$ be the inclusion, and let

$$i^* : \mathcal{A}^{1,1}(\hat{X}) \rightarrow \mathcal{A}^{1,1}(F)$$

be the pull-back. Since $\pi \circ i$ is constant we see that

$$\partial\bar{\partial}(i^*\hat{f}) = i^*(\pi^*\omega) = 0.$$

Thus if $\mu : F' \rightarrow F$ is a desingularisation, we have $\partial\bar{\partial}(\mu^*i^*\hat{f}) = 0$. Thus $\mu^*i^*\hat{f}$ is harmonic, hence constant. Since π has connected fibres, \hat{f} is constant along the π -fibres. Thus $\hat{f} = \pi^*(f)$ with a function $f \in \mathcal{A}^0(X)^3$. Consequently we have $\omega = \partial\bar{\partial}f$, hence $\{\omega\} = 0$. \square

3.4. Definition. *Let X be a normal compact complex space. Then $N_{\mathbb{C}}^1(X)$ is the vector space of d -closed forms $\omega \in \mathcal{A}^{1,1}(X)$ modulo the following equivalence relation:*

$$\omega_1 \equiv \omega_2 \text{ if and only if } T(\omega_1) = T(\omega_2)$$

for all $T \in \mathcal{D}^{n-1,n-1}(X)$ with $dT = 0$. Given a d -closed form $\omega \in \mathcal{A}^{1,1}(X)$, its $N_{\mathbb{C}}^1(X)$ -class is denoted by $[\omega]$.

We denote by $N^1(X) \subset N_{\mathbb{C}}^1(X)$ the space generated by the d -closed forms $\omega \in \mathcal{A}_{\mathbb{R}}^{1,1}(X)$.

Note that for X projective, the space $N^1(X) = N_{\mathbb{R}}^1(X)$ is usually defined differently, namely as the \mathbb{R} -vector space generated by the classes of irreducible divisors. Since we are dealing with general compact complex spaces, this space is often too small to be useful.

Note also that if $\pi : \hat{X} \rightarrow X$ is a desingularisation, we have a well-defined pull-back

$$\pi^* : N_{\mathbb{C}}^1(X) \rightarrow N_{\mathbb{C}}^1(\hat{X}), [\omega] \mapsto [\pi^*\omega].$$

Indeed if ω is a d -closed $(1,1)$ -form on X such that $T(\omega) = 0$ for all closed currents on X , then for every closed current \hat{T} on \hat{X} we have

$$\hat{T}(\pi^*\omega) = \pi_*(\hat{T})(\omega) = 0,$$

since $\pi_*(\hat{T})$ is a closed current on X .

3.5. Lemma. *Let X be a normal compact complex space of dimension n in class \mathcal{C} . Then the natural map*

$$N_{\mathbb{C}}^1(X) \rightarrow H_{\partial\bar{\partial}}^{1,1}(X), [\omega] \mapsto \{\omega\}$$

³This is well-known to experts, for lack of reference we give a sketch of the proof. It is obvious that f is continuous and smooth outside the singular locus; the only issue is to show that f is smooth at the singular points of X . Fix a point $x_0 \in X_{\text{sing}}$ and an open neighborhood $x_0 \in U \subset X$ admitting a closed embedding $U \subset V$ into an open set $V \subset \mathbb{C}^N$. We want to show that there exists a C^∞ -function $F : V \rightarrow \mathbb{C}$ such that $f = F|_U$. Let $\sigma : \hat{V} \rightarrow V$ be a sequence of blow-ups along smooth centres that is an embedded resolution $\hat{U} \rightarrow U$ and such that the exceptional locus is a normal crossings divisor E meeting \hat{U} transversally. We can suppose without loss of generality that $\hat{U} \rightarrow U$ factors through the resolution $\pi^{-1}(U) \rightarrow U$. Pulling f back to \hat{U} we can suppose without loss of generality that $\hat{U} = \pi^{-1}(U)$. We extend \hat{f} to $\hat{U} \cup E$ by $\hat{f}|_{\sigma^{-1}(x)} = \text{const.} = \hat{f}(y)$ with $y \in \pi^{-1}(x)$ an arbitrary point. The extension is a differentiable function on $\hat{U} \cup E$ (i.e. the restriction to every smooth stratum is differentiable; apply e.g. [SS92]), so it extends to a differentiable function $\hat{F} : \hat{V} \rightarrow \mathbb{C}$ that is constant along the π -fibres such that $\hat{F}|_{\hat{U}} = \hat{f}$. We have $\hat{F} = \sigma^*F$ with F a function on V . Since σ is a composition of smooth blow-ups, it is elementary to see that F is C^∞ .

is well-defined and an isomorphism.

Proof. Let $\pi : \hat{X} \rightarrow X$ be a desingularisation such that \hat{X} is a compact Kähler manifold. By classical Hodge theory the $\partial\bar{\partial}$ -group $H_{\partial\bar{\partial}}^{1,1}(\hat{X})$ is canonically isomorphic to the Dolbeault group $H^{1,1}(\hat{X})$ which is Poincaré dual to $H^{n-1,n-1}(\hat{X})$. Moreover every cohomology class in $H^{n-1,n-1}(\hat{X})$ is represented by a current of bidegree $(n-1, n-1)$ [GH78, p.382ff].

Let now ω be a d -closed $(1,1)$ -form on X such that $T(\omega) = 0$ for all closed currents of bidegree $(n-1, n-1)$ on X . Then $\pi^*\omega$ is a d -closed $(1,1)$ -form on \hat{X} such that $\hat{T}(\pi^*\omega) = 0$ for all closed currents of bidegree $(n-1, n-1)$ on \hat{X} . By the previous considerations, the $\partial\bar{\partial}$ -class of $\pi^*\omega$ is zero, so we have

$$\pi^*\omega = \partial\bar{\partial}\hat{f},$$

with $\hat{f} \in \mathcal{A}^0(\hat{X})$. Thus we have $\{\pi^*\omega\} = 0$, hence $\{\omega\} = 0$ by Lemma 3.3.

This shows that the natural map $N_{\mathbb{C}}^1(X) \rightarrow H_{\partial\bar{\partial}}^{1,1}(X)$ is well-defined; it is clearly surjective since both spaces are quotients of

$$\{\omega \in \mathcal{A}^{1,1}(X) \mid d\omega = 0\}.$$

The map is also injective : if $\omega = \partial\bar{\partial}f = d(\bar{\partial}f)$, then

$$T(\omega) = T(d(\bar{\partial}f)) = dT(\bar{\partial}f) = 0$$

for every d -closed current T of bidegree $(n-1, n-1)$ on X . \square

3.6. Definition. Let X be a normal compact complex space. Then $N_1(X)$ is the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \sim T_2$ if and only if

$$T_1(\eta) = T_2(\eta)$$

for all real closed $(1,1)$ -forms η .

Let ω be a real closed $(1,1)$ -form on X . Then we can define

$$\lambda_\omega \in N_1(X)^*, [T] \mapsto T(\omega).$$

If $T(\omega) = 0$ for all closed currents T of bidimension $(1,1)$, we have $\lambda_\omega = 0$.

Thus we have a well-defined canonical map

$$\Phi : N^1(X) \rightarrow N_1(X)^*, [\omega] \mapsto \lambda_\omega.$$

3.7. Proposition. Let X be a normal compact complex space of dimension n in class \mathcal{C} . Then the canonical map Φ is an isomorphism. In particular $N_1(X)$ is finite-dimensional. Moreover given a linear map $\lambda : N_1(X) \rightarrow \mathbb{R}$, there exists a real closed $(1,1)$ -form ω such that

$$\lambda([T]) = T(\omega).$$

If $\mu : X' \rightarrow X$ is a bimeromorphic map, the natural linear map

$$\mu_* : N_1(X') \rightarrow N_1(X)$$

is surjective.

Proof. By Lemmas 3.3 and 3.5 the vector space $N^1(X)$ is finite-dimensional. Let ω be a real closed $(1,1)$ -form such that $\Phi([\omega]) = 0$. Then $T(\omega) = 0$ for all closed currents T of bidegree $(n-1, n-1)$. By definition of $N^1(X)$ this gives $[\omega] = 0$. Thus Φ is injective.

Let us now show that Φ is surjective. Since $N^1(X)$ is finite-dimensional, it suffices to show that $\dim N_1(X) \leq \dim N^1(X)$. However this is clear, since we can prove as above that the natural map

$$\psi : N_1(X) \rightarrow N^1(X)^*, [T] \mapsto \psi([T])([\omega]) = T(\omega)$$

is well-defined and injective.

For the proof of the second statement, let $\pi : \hat{X} \rightarrow X$ be a desingularisation of X that factors through μ . By Lemmas 3.3 and 3.5 the pull-back

$$\pi^* : H_{\partial\bar{\partial}}^{1,1}(X) \simeq N_{\mathbb{C}}^1(X) \rightarrow H_{\partial\bar{\partial}}^{1,1}(\hat{X}) \simeq N_{\mathbb{C}}^1(\hat{X})$$

is injective, so by duality the push-forward $\pi_* : N_1(\hat{X}) \rightarrow N_1(X)$ is surjective. By functoriality of the push-forward this shows that

$$\mu_* : N_1(X') \rightarrow N_1(X)$$

is also surjective. \square

3.8. Definition. Let X be a normal compact complex space of dimension n in class \mathcal{C} . We define $\overline{NA}(X) \subset N_1(X)$ as the cone generated by the positive closed currents of bidimension $(1,1)$.

3.9. Remark. Given an irreducible curve $C \subset X$ we associate the current of integration T_C , and define the Mori cone $\overline{NE}(X) \subset N_1(X)$ as the closure of the cone generated by the currents T_C . We clearly have an inclusion

$$\overline{NE}(X) \subset \overline{NA}(X).$$

Let us note that this inclusion can be strict even if X is a projective manifold: in this case $\overline{NE}(X)$ generates a vector space of dimension equal to the Picard number $\rho(X)$, while $\overline{NA}(X)$ generates $H^{n-1,n-1}(X)$.

3.10. Proposition. Let X be a normal compact complex space in class \mathcal{C} . Let $\mu : X' \rightarrow X$ be a bimeromorphic map. Then we have

$$\mu_*(\overline{NA}(X')) = \overline{NA}(X).$$

Proof. Let $\pi : \hat{X} \rightarrow X$ be a desingularisation of X that factors through μ and such that \hat{X} is a compact Kähler manifold. By the functoriality of the push-forward it is sufficient to prove that

$$\pi_*(\overline{NA}(\hat{X})) = \overline{NA}(X).$$

We clearly have $\pi_*(\overline{NA}(\hat{X})) \subset \overline{NA}(X)$. Arguing by contradiction we assume that $\pi_*(\overline{NA}(\hat{X}))$ is a proper subcone of $\overline{NA}(X)$. The space $N_1(X)$ being finite-dimensional, there exists a linear map

$$\lambda : N_1(X) \rightarrow \mathbb{R}$$

which is non-negative on $\pi_*(\overline{NA}(\hat{X}))$, and $\lambda(T_0) < 0$ for some $T_0 \in \overline{NA}(X)$. By Proposition 3.7, there exists a real closed $(1,1)$ -form $\eta \in \mathcal{A}^{1,1}(X)$ such that $\lambda([T]) = T(\eta)$ for all $[T] \in N_1(X)$. We define

$$\hat{\lambda} : N_1(\hat{X}) \rightarrow \mathbb{R}$$

by $\hat{\lambda}([\hat{T}]) = \hat{T}(\pi^*\eta)$. Then we have

$$\hat{\lambda}([\hat{T}]) = \hat{T}(\pi^*(\eta)) = \pi_*(\hat{T})(\eta) \geq 0$$

for all positive closed currents \hat{T} on \hat{X} , so $\hat{\lambda}$ is non-negative on $\overline{NA}(\hat{X})$. Since $\overline{NA}(\hat{X})$ is dual to the Kähler cone (a well-known consequence of [DP04, Cor.0.3]; see e.g. [OP04, Prop.1.8]), we conclude that $\{\pi^*(\eta)\}$ is a nef class. Thus by [Pau98, Thm.1], [DP04, Prop.0.6] the class $\{\eta\}$ is nef in the sense of [Pau98, Def.3]. Using the isomorphism in Lemma 3.5 we see that $[\eta] = \{\eta\}$ is non-negative on $\overline{NA}(X)$, a contradiction. \square

3.11. Definition. Let X be a normal compact complex space in class \mathcal{C} . We denote by $\text{Nef}(X) \subset N^1(X)$ the cone generated by cohomology classes which are nef in the sense of [Pau98, Defn.3].

3.12. Remark. If X is a normal compact Kähler space we can also consider the open cone \mathcal{K} generated by the classes of Kähler forms. In this case we know⁴ that

$$\text{Nef}(X) = \overline{\mathcal{K}}.$$

3.13. Proposition. Let X be a normal compact complex space in class \mathcal{C} . Then the cones $\text{Nef}(X)$ and $\overline{NA}(X)$ are dual via the canonical isomorphism $\Phi : N^1(X) \rightarrow N_1(X)^*$, constructed in Proposition 3.7.

Proof. For a closed convex cone V in some finite-dimensional real vector space we have $V = V^{**}$ [Deb01, Lemma 6.7], so it is sufficient to prove that

$$\text{Nef}(X) = \overline{NA}(X)^*.$$

It is clear that $\text{Nef}(X) \subset \overline{NA}(X)^*$ so we are left to prove the other inclusion. We consider a linear form $\lambda : N_1(X) \rightarrow \mathbb{R}$ such that $\lambda(T) \geq 0$ for all $[T] \in \overline{NA}(X)$. Choose a closed $(1,1)$ -form ω such that $T(\omega) = \lambda([T])$ then we want to show that $[\omega] \in \text{Nef}(X)$.

Let $\pi : \hat{X} \rightarrow X$ be a desingularisation such that \hat{X} is a compact Kähler manifold, then $\pi^*\omega$ defines a linear form on $N_1(\hat{X})$ which is non-negative on $\overline{NA}(\hat{X})$. Thus $[\pi^*\omega]$ is a nef class by [DP04, Cor.0.3], so the class $[\omega]$ is nef by [Pau98, Thm.1], [DP04, Prop.0.6]. \square

In the Kähler case $\overline{NA}(X)$ is usually defined right away as the dual of the Kähler cone.

⁴The statement in [Dem92, Prop.6.1.iii)] is for compact manifolds, but the proof works in the singular setting.

3.14. Corollary. *Let X be a normal compact Kähler space. Suppose that $[\eta] \in N^1(X)$ is strictly positive on $\overline{NA}(X) \setminus 0$. Then $[\eta]$ is a Kähler class, i.e. can be represented by a Kähler form ω .*

Proof. By Proposition 3.13 we see that $[\eta]$ lies in the interior of the cone $\text{Nef}(X)$. Since X is Kähler we have $\text{Nef}(X) = \overline{\mathcal{K}}$ (cf. Remark 3.12), so $[\eta]$ is a Kähler class. \square

3.15. Theorem. *Let X be an irreducible and reduced compact complex space of dimension n in class \mathcal{C} . Let $\eta \in \mathcal{A}^{1,1}(X)$ be a closed form such that $T(\eta) > 0$ for all positive closed currents $T \in \mathcal{D}^{n-1,n-1}(X) \setminus \{0\}$. Then $\{\eta\}$ contains a Kähler form, in particular X is a Kähler space.*

Proof. We will argue by induction on the dimension, the case $n = 0$ being clear.

Let $\pi : \hat{X} \rightarrow X$ be a desingularisation such that \hat{X} is a compact Kähler manifold and such that π is a projective morphism with exceptional locus a simple normal crossings divisor. Then there exists an effective π -exceptional divisor D on \hat{X} such that $-D$ is π -ample. Moreover for large N the class

$$\{N\pi^*(\eta) - D\}$$

is strictly positive on $\overline{NA}(\hat{X})$. By [DP04, Cor.0.4] there exists a Kähler form $\hat{\omega}$ on \hat{X} such that

$$\{N\pi^*(\eta) - D\} = \{\hat{\omega}\}.$$

Thus we can write

$$\hat{\omega} = N\pi^*(\eta) - T_D + \partial\bar{\partial}\hat{f}$$

where $\hat{f} \in \mathcal{A}^0(\hat{X})$ and T_D the current of integration associated with D . Applying π_* it follows that

$$\pi_*(\hat{\omega}) = N\eta + \partial\bar{\partial}R$$

with $R = \pi_*(\hat{f})$, which by definition means that $\{\pi_*(\hat{\omega})\} = \{N\eta\}$ (cf. Remark 3.2). By [DP04, Prop.0.6] the class $\{\pi_*(\hat{\omega})\} = \{N\eta\}$ of the Kähler current $\pi_*(\hat{\omega})$ contains a Kähler form if and only if $\{\pi_*(\hat{\omega})\}|_Z$ is a Kähler class for every irreducible component Z of the Lelong level sets. Let $E \subset X$ be the image of the π -exceptional locus. The Kähler current $\pi_*(\omega)$ is smooth outside E , hence any Lelong level set Z is contained in E . To verify that $\{\eta\}_Z = \{i_Z^*(\eta)\}$ contains a Kähler class, we are going to apply the induction hypothesis and only need to show that $T'(i_Z^*(\eta)) > 0$ for all closed positive currents T' on Z . However, $(i_Z)_*(T')$ is a non-zero positive closed current on X , hence $T'(i_Z^*(\eta)) = (i_Z)_*(T')(\eta) > 0$. \square

3.16. Theorem. *Let X be a normal compact threefold in class \mathcal{C} with only isolated singularities. Let $\eta \in \mathcal{A}^{1,1}(X)$ be a closed real $(1,1)$ -form such that $T(\eta) > 0$ for all $[T] \in \overline{NA}(X) \setminus 0$. Suppose that for every irreducible curve $C \subset X$ we have $[C] \neq 0$ in $N_1(X)$. Then $\{\eta\}$ is represented by a Kähler class, in particular X is Kähler.*

Proof. Step 1. Suppose that X is smooth. We argue by contradiction. By Theorem 3.15 there exists a positive closed current $T \neq 0$ such that $T(\eta) = 0$. By our assumption this implies that $[T] = 0$ in $N_1(X)$.

Let $\pi : \hat{X} \rightarrow X$ be a modification of X such that \hat{X} is a compact Kähler manifold. Let $S \subset X$ be the image of the π -exceptional locus. We write $S = \bigcup C_j \cup A$ with C_j the irreducible components of dimension 1 and A a finite set. We consider the positive closed currents $\chi_S T$ and $\chi_{X \setminus S} T$. Since η is non-negative on $\overline{NA}(X)$ the decomposition

$$T = \chi_S T + \chi_{X \setminus S} T$$

implies that $\chi_S T(\eta) = \chi_{X \setminus S} T(\eta) = 0$. By our assumption this implies $[\chi_S T] = [\chi_{X \setminus S} T] = 0$. By a theorem of Siu [Siu74] we have

$$\chi_S T = \sum \alpha_j T_{C_j}$$

with $\alpha_j \geq 0$ and T_{C_j} denoting the current of integration over C_j . Since $[\chi_S T] = 0$ and every curve class is non-zero, we conclude that $\alpha_j = 0$ for all j . Thus we have $\chi_S T = 0$.

It remains to prove that $\chi_{X \setminus S} T = 0$. Fix a Kähler form $\hat{\omega}$ on \hat{X} and consider the positive closed current $R = \pi_*(\hat{\omega})$. Choose smooth closed $(1,1)$ -forms ψ_ε which converge weakly to R . Choose a sequence φ_j of non-negative \mathcal{C}^∞ functions with compact support in $X \setminus S$ converging increasingly to $\chi_{X \setminus S}$. Then $\varphi_j R$ is a positive (non-closed) smooth $(1,1)$ -form on X , hence $(\chi_{X \setminus S} T)(\varphi_j R) \geq 0$. On the other hand,

$$\begin{aligned} \lim_{j \rightarrow \infty} (\chi_{X \setminus S} T)(\varphi_j R) &= \lim_{j \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} (\chi_{X \setminus S} T)(\varphi_j \psi_\varepsilon) = \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} (\chi_{X \setminus S} T)(\varphi_j \psi_\varepsilon) = \lim_{\varepsilon \rightarrow 0} (\chi_{X \setminus S} T)(\psi_\varepsilon). \end{aligned}$$

Since $[\chi_{X \setminus S} T] = 0$ we have in particular $\lim_{\varepsilon \rightarrow 0} (\chi_{X \setminus S} T)(\psi_\varepsilon) = 0$. Since the sequence $(\chi_{X \setminus S} T)(\varphi_j R)$ is increasing we see that $(\chi_{X \setminus S} T)(\varphi_j R) = 0$ for all j . Hence the support of $(\chi_{X \setminus S} T)$ is contained in S , hence $\chi_{X \setminus S} T = 0$.

Step 2. Reduction to the smooth case. Let $\mu : \hat{X} \rightarrow X$ be a desingularisation, and let E be an effective μ -exceptional divisor such that $-E$ is μ -ample. Let $\hat{C} \subset \hat{X}$ be an irreducible curve. If \hat{C} is μ -exceptional, then $-E \cdot \hat{C} > 0$, so $[\hat{C}] \neq 0$ in $N_1(\hat{X})$. If \hat{C} is not μ -exceptional, then we have

$$\mu_*([\hat{C}]) = [\mu(\hat{C})] \neq 0,$$

so $[\hat{C}] \neq 0$ in $N_1(\hat{X})$. We claim that there exists a number $n \in \mathbb{N}$ such that

$$T(\mu^* \eta - \frac{1}{n} E) > 0$$

for all $[T] \in \overline{NA}(\hat{X}) \setminus 0$. Assuming this for the time being, let us see how to conclude: by Step 1 we know that \hat{X} is a Kähler manifold, so there exists a Kähler form $\hat{\omega}$ on \hat{X} . The push-forward $\mu_*(\hat{\omega})$ is a Kähler current on X and the Lelong level sets of $\mu_*(\hat{\omega})$ are contained in the singular locus of X . Since X

has isolated singularities, we see that $\{\mu_*(\hat{\omega})\}|_Z$ is a Kähler class for every Z an irreducible component of the Lelong level sets. By [DP04, Prop.0.6] this shows that $\{\mu_*(\hat{\omega})\}$ contains a Kähler class.

Proof of the claim. Fixing a hermitian metric h on \hat{X} , we see that it is sufficient to prove that

$$T(\mu^*\eta - \frac{1}{n}E) > 0$$

for all $[T] \in (\overline{NA}(\hat{X}) \cap H)$ where H is the hypersurface of classes having “norm” one with respect to the hermitian metric h . Note that $\overline{NA}(\hat{X}) \cap H$ is a compact set, in particular the function

$$e : \overline{NA}(\hat{X}) \cap H \rightarrow \mathbb{R}, \quad T \mapsto T(E)$$

is bounded from above. Arguing by contradiction we assume that for all $n \in \mathbb{N}$ there exists a class $[T_n] \in (\overline{NA}(\hat{X}) \cap H)$, represented by a positive closed current T_n , such that

$$(1) \quad T_n(\mu^*\eta - \frac{1}{n}E) \leq 0.$$

Since $\overline{NA}(\hat{X}) \cap H$ is compact, we can suppose, up to taking subsequences, that the sequence $[T_n]$ converges to a limit $[T_\infty] \neq 0$, represented by a positive closed current T_∞ . Since $T_n(E)$ is bounded, the equation (1) simplifies to

$$T_\infty(\mu^*\eta) \leq 0.$$

Since η is positive on $\overline{NA}(X) \setminus \{0\}$, this implies $[\mu_*(T_\infty)] = 0$. Since $-E$ is μ -ample we conclude that $T_\infty(E) < 0$. Yet by continuity this implies that $T_n(E) < 0$ for $n \in \mathbb{N}$ sufficiently large. However, (1) is equivalent to

$$T_n(E) \geq n T_n(\mu^*\eta) \geq 0,$$

a contradiction. □

The importance of the dual Kähler cone $\overline{NA}(X)$ in our context lies in the following

3.17. Definition. *Let X be a normal \mathbb{Q} -factorial compact Kähler space with at most terminal singularities, and let $\mathbb{R}^+[\Gamma_i]$ be a K_X -negative extremal ray in $\overline{NA}(X)$. A contraction of the extremal ray $\mathbb{R}^+[\Gamma_i]$ is a morphism $\varphi : X \rightarrow Y$ onto a normal compact Kähler space such that $-K_X$ is φ -ample and a curve $C \subset X$ is contracted if and only if $[C] \in \mathbb{R}^+[\Gamma_i]$.*

4. SUBVARIETIES OF KÄHLER THREEFOLDS

4.A. Remarks on adjunction. Let X be normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities. Let $S \subset X$ be a prime divisor, i.e. an irreducible and reduced compact surface. Let $m \in \mathbb{N}$ be the

smallest positive integer such that both mK_X and mS are Cartier divisors on X . Then the canonical divisor $K_S \in \text{Pic}(S) \otimes \mathbb{Q}$ is defined by

$$K_S := \frac{1}{m}(mK_X + mS)|_S.$$

Since X is smooth in codimension two, there exist at most finitely many points $\{p_1, \dots, p_q\}$ where K_X and S are not Cartier. Thus by the adjunction formula K_S is isomorphic to the dualising sheaf ω_S on $S \setminus \{p_1, \dots, p_q\}$.

Let now $\nu : \tilde{S} \rightarrow S$ be the normalisation. Then we have

$$(2) \quad K_{\tilde{S}} \sim_{\mathbb{Q}} \nu^* K_S - N,$$

where N is an effective Weil divisor defined by the conductor ideal. Indeed this formula holds by [Rei94] for the dualising sheaves. Since $\mathcal{O}_{\tilde{S}}(\nu^* K_S)$ is isomorphic to $\nu^* \omega_S$ on the complement of $\nu^{-1}(p_1, \dots, p_q)$, the formula holds for the canonical divisors.

Let $\mu : \hat{S} \rightarrow \tilde{S}$ be the minimal resolution of the normal surface S , then we have

$$K_{\hat{S}} \sim_{\mathbb{Q}} \mu^* K_{\tilde{S}} - N',$$

where N' is an effective μ -exceptional divisor [Sak84, 4.1]. Thus if $\pi : \hat{S} \rightarrow S$ is the composition $\nu \circ \mu$, there exists an effective, canonically defined divisor $E \subset \hat{S}$ such that

$$(3) \quad K_{\hat{S}} \sim_{\mathbb{Q}} \pi^* K_S - E.$$

Let $C \subset S$ be a curve such that $C \not\subset S_{\text{sing}}$. Then the morphism π is an isomorphism in the generic point of C , and we can define the strict transform $\hat{C} \subset \hat{S}$ as the closure of $C \setminus S_{\text{sing}}$. Since \hat{C} is an (irreducible) curve that is not contained in the divisor N defined by the conductor, we have $\hat{C} \not\subset E$. By the projection formula and (3) we obtain

$$(4) \quad K_{\hat{S}} \cdot \hat{C} \leq K_S \cdot C.$$

4.B. Divisorial Zariski decomposition for K_X . Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities. If K_X is not pseudoeffective we know by a theorem of Brunella [Bru06] that X is uniruled. In particular we have a covering family of rational curves such that $K_X \cdot C < 0$.

From now on suppose that K_X is pseudoeffective. By [Bou04, Thm.3.12] we have a divisorial Zariski decomposition⁵

$$(5) \quad K_X = \sum_{j=1}^r \lambda_j S_j + N(K_X),$$

where the S_j are integral surfaces in X , the coefficients $\lambda_j \in \mathbb{R}^+$ and $N(K_X)$ is a pseudoeffective \mathbb{R} -divisor which is nef in codimension one [Bou04, Prop.2.4], that is for every surface $S \subset X$ the restriction $N(K_X)|_S$ is pseudoeffective.

4.1. Lemma. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let S be a surface such that $K_X|_S$ is not pseudoeffective. Then S is one of the surfaces S_j in the divisorial Zariski decomposition (5) of K_X . Moreover $S = S_j$ is projective and uniruled.*

Proof. Suppose first that $S \neq S_j$ for every $j \in \{1, \dots, r\}$. Then we have

$$K_X|_S = \sum_{j=1}^r \lambda_j (S \cap S_j) + N(K_X)|_S.$$

Since $N(K_X)|_S$ is pseudoeffective, we see that $K_X|_S$ is pseudoeffective, a contradiction. Thus we have $S = S_j$ for some j and (up to renumbering) we can suppose that $S = S_1$. We have

$$S = S_1 = \frac{1}{\lambda_1} K_X - \frac{1}{\lambda_1} \left(\sum_{j=2}^r \lambda_j S_j + N(K_X) \right),$$

so by adjunction

$$K_S = (K_X + S)|_S = \frac{\lambda_1 + 1}{\lambda_1} K_X|_S - \frac{1}{\lambda_1} \left(\sum_{j=2}^r \lambda_j (S_j \cap S) + N(K_X)|_S \right).$$

By hypothesis $K_X|_S$ is not pseudoeffective and $-\frac{1}{\lambda_1} \left(\sum_{j=2}^r \lambda_j (S_j \cap S) + N(K_X)|_S \right) + N(K_X)|_S$ is anti-pseudoeffective. Thus K_S is not pseudoeffective.

Let now $\pi : \hat{S} \rightarrow S$ be the composition of normalisation and minimal resolution of the normalised surface, then by (3) there exists an effective divisor E such that

$$K_{\hat{S}} \sim_{\mathbb{Q}} \pi^* K_S - E.$$

⁵The statements in [Bou04] are for complex compact manifolds, but generalise immediately to our situation: take $\mu : X' \rightarrow X$ a desingularisation, and let $m \in \mathbb{N}$ be the Cartier index of K_X . Then $\mu^*(mK_X)$ is a pseudoeffective line bundle with divisorial Zariski decomposition $\mu^*(mK_X) = \sum \eta_j S'_j + N(mK_X)'$. The decomposition of K_X is defined by the push-forwards $\mu_* \left(\frac{1}{m} \sum \eta_j S'_j \right)$ and $\mu_* \left(\frac{1}{m} N(mK_X)' \right)$. Since a prime divisor $D \subset X$ is not contained in the singular locus of X , the decomposition has the stated properties.

Thus $K_{\hat{S}}$ is not pseudoeffective, in particular $\kappa(\hat{S}) = -\infty$. By Remark 2.3 the surface \hat{S} is Kähler, so it follows from the Castelnuovo-Kodaira classification that \hat{S} is covered by rational curves, in particular it is a projective surface [BHPVdV04]. \square

4.2. Corollary. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then K_X is nef if and only if*

$$K_X \cdot C \geq 0$$

for every curve $C \subset X$.

Proof. One implication is trivial. Suppose now that K_X is nef on all curves C . We will argue by contradiction and suppose that K_X is not nef. Since K_X is pseudoeffective and the restriction to every curve is nef, there exists by [Bou04, Prop.3.4] an irreducible surface $S \subset X$ such that $K_X|_S$ is not pseudoeffective. By Lemma 4.1 the surface S is projective, and the divisor $K_X|_S$ is not pseudoeffective, so there exists a covering family of curves $C_t \subset S$ such that

$$K_X \cdot C_t = K_X|_S \cdot C_t < 0,$$

a contradiction. \square

4.C. Very rigid curves.

4.3. Definition. *Let X be a normal \mathbb{Q} -factorial Kähler threefold with at most terminal singularities. We say that a curve C is very rigid if*

$$\dim_{mC} \text{Chow}(X) = 0$$

for all $m > 0$.

4.4. Remark. Kollár [Kol91a, Defn.4.1] defines a curve $C \subset X$ to be very rigid if there is no flat family of one dimensional subschemes $D_t \subset X$ parametrised by a pointed disc $0 \in \Delta$ such that

- $\text{Supp } D_0 = C$,
- $\text{Supp } D_t \not\subset C$ for $t \neq 0$, and
- D_t is purely one dimensional for $t \neq 0$.

A curve that is very rigid in the sense of Definition 4.3, is also very rigid in the sense of Kollár: using the natural map from the normalisation of the Douady space $\mathcal{H}(X)$ to the cycle space $\text{Chow}(X)$ [Kol96, Thm.6.3], a family $(D_t)_{t \in \Delta}$ defines a positive-dimensional subset of $\text{Chow}(X)$ through a point corresponding to mC for some $m > 0$.

4.5. Theorem. *Let X be a normal \mathbb{Q} -factorial Kähler threefold with at most terminal singularities. Let $C \subset X$ be an irreducible very rigid curve such that $K_X \cdot C < 0$. Then C is a rational curve and*

$$K_X \cdot C \geq -1.$$

For the proof of this theorem we follow the arguments of Kollár [Kol91a], with some simplifications due to our special situation. It is actually not necessary to work in the algebraic category, all arguments work in the analytic category as well.

Proof. Step 1. If $K_X \cdot C < -1$, then C is not very rigid. Let \hat{X} be the formal completion of X along C . Choose m minimal such that mK_X is Cartier. Since $-mK_X|_C$ is ample, the restriction $-mK_X|_{X_k}$ to the k -th infinitesimal neighborhood X_k of C in X is ample [Har70, Prop.4.2]. Thus there are projective schemes Z_k such that $(Z_k)_{\text{an}} \simeq X_k$. Then the formal scheme arising as inverse limit Z of the system (Z_k) is the algebraization of \hat{X} .

Following the notations of Kollár [Kol91a, Constr.1.2], choose $d \in \mathbb{Z}$ with $0 \leq d < m$ such that

$$d \equiv -mK_X \cdot C \pmod{m}.$$

Choose a general Cartier divisor D on \hat{X} such that D meets C in a single point p_0 transversally. By the general choice of D this will be a smooth point of \hat{X} . Therefore the pull-back

$$\nu^* \mathcal{O}_{\hat{X}}(-dD - mK_{\hat{X}})$$

to the normalisation $\nu : \tilde{C} \rightarrow C$ is divisible by m in $\text{Pic}(\tilde{C})$. We claim that

$$\mathcal{O}_{\hat{X}}(-dD - mK_{\hat{X}})$$

is divisible by m in $\text{Pic}(\hat{X})$. To see this observe that for each $k \in \mathbb{N}$ the kernel of the restriction

$$\text{Pic}(X_{k+1}) \rightarrow \text{Pic}(X_k)$$

is an extension by an additive and a multiplicative group (use the exponential sequence for X_k , cp. [Kol91a, 1.2.1] and [Gro62, no.232, Prop.6.5]). These groups are divisible, so

$$\mathcal{O}_{X_k}(-dD - mK_{\hat{X}})$$

is divisible by m in $\text{Pic}(X_k)$, hence $\mathcal{O}_{\hat{X}}(-dD - mK_{\hat{X}})$ is divisible by m in $\text{Pic}(\hat{X})$ [Har77, II, Ex.9.6].

Write

$$\mathcal{O}_{\hat{X}}(-dD - mK_{\hat{X}}) \simeq N^{\otimes m}$$

with a line bundle N on \hat{X} . We obtain a section

$$s \in H^0(\hat{X}, (N^*)^{\otimes m} \otimes \mathcal{O}_{\hat{X}}(-mK_{\hat{X}}))$$

via

$$N^{\otimes m} \otimes \mathcal{O}_{\hat{X}}(mK_{\hat{X}}) \simeq \mathcal{O}_{\hat{X}}(-dD) \rightarrow \mathcal{O}_{\hat{X}}.$$

Let

$$p : \overline{X} \rightarrow \hat{X}$$

be the normalisation of $\hat{X}[s^{\frac{1}{m}}]$. By construction p ramifies exactly over D and the non-Gorenstein points of \hat{X} . Moreover \overline{X} is Gorenstein and has at most terminal singularities by [KM98, 5.21(4)]. We introduce the number

$$e := \frac{m}{\gcd(d, m)}.$$

Then p has ramification order e over D yielding the equation

$$(6) \quad K_{\overline{X}} = p^*(K_{\hat{X}} + (1 - \frac{1}{e})D).$$

Let $\overline{C} \subset p^{-1}(C)$ be an irreducible component (with reduced structure). We claim that if $K_{\hat{X}} \cdot C < -1$, then $K_{\hat{X}} \cdot \overline{C} \leq -2$. Assuming this for the time being, let us see how to conclude: since

$$K_{\hat{X}} \cdot C = K_X \cdot C < -1,$$

we know by the claim that $K_{\overline{X}} \cdot \overline{C} \leq -2$. But now \overline{X} is a threefold with Gorenstein terminal singularities, so it has at most lci singularities. Therefore \overline{C} deforms in \overline{X} by [Kol96, Thm.1.15, Rem.1.17]. Hence some multiple of C deforms in \hat{X} .

Proof of the claim. Since $K_{\hat{X}} \cdot C$ is an integer multiple of $\frac{1}{e}$, our assumption implies $K_{\hat{X}} \cdot C \leq -1 - \frac{1}{e}$. Since $D \cdot C = 1$ we obtain

$$K_{\hat{X}} \cdot C + (1 - \frac{1}{e})D \cdot C \leq -\frac{2}{e}$$

The order of ramification for every point p_0 over D is equal to e , so we have $\deg(\overline{C}/C) \geq e$. Thus the ramification formula (6) and the preceding inequality gives

$$K_{\overline{X}} \cdot \overline{C} = \deg(\overline{C}/C) \left(K_{\hat{X}} \cdot C + (1 - \frac{1}{e})D \cdot C \right) \leq -2.$$

This proves the claim.

Step 2. Very rigid curves K_X -negative curves are rational. If C is irrational and $K_X \cdot C < 0$, then there are finite étale covers

$$h : C' \rightarrow C$$

of arbitrary large degree. Choose an open neighborhood U of C in X such that C is a deformation retract of U . Then $\pi_1(U) \simeq \pi_1(C)$ and therefore we obtain a finite étale cover $g : U' \rightarrow U$ such that $C' \subset U'$ and $h = g|_{C'}$. Hence $K_{U'} \cdot C'$ gets arbitrarily negative, in particular smaller than -1 . By Step 1 the curve C' is not very rigid, hence C is not very rigid. \square

5. BEND AND BREAK

For the convenience of the reader we recall two results from the deformation theory of curves.

5.1. Theorem. [Kol96, Thm.1.15] *Let S be a smooth complex surface, and let $C \subset S$ be a curve. Then we have*

$$\dim_C \text{Chow}(S) \geq -K_S \cdot C + \chi(\mathcal{O}_{\tilde{C}}),$$

where $\tilde{C} \rightarrow C$ is the normalisation.

Recall that terminal Gorenstein threefold singularities are hypersurfaces singularities [Rei80], [KM98, Cor.5.38] so we have:

5.2. Theorem. [Kol96, Thm.1.15] *Let X be a normal \mathbb{Q} -factorial complex threefold with at most terminal Gorenstein singularities. Let $C \subset X$ be a curve. Then we have*

$$\dim_C \text{Chow}(X) \geq -K_X \cdot C.$$

5.3. Definition. *Let X be a complex space and Z an effective 1-cycle on X . A deformation family of Z is a family of cycles $(Z_t)_{t \in T}$ where T is an irreducible component of the cycle space $\text{Chow}(X)$ that contains the point corresponding to Z . We say that the family of deformations splits if there exists a $t_0 \in T$ such that the cycle*

$$Z_{t_0} = \sum \alpha_l C_l$$

has reducible support or $Z_{t_0} = \alpha_1 C_1$ with $\alpha_1 \geq 2$.

The following elementary lemma will be used several times.

5.4. Lemma. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let $C \subset X$ be a curve such that $K_X \cdot C < 0$ and $\dim_C \text{Chow}(X) > 0$.*

Then there exists a unique surface S_j from the divisorial Zariski decomposition (5) such that C and its deformations are contained in the surface S_j . Moreover we have

$$(7) \quad K_{S_j} \cdot C < K_X \cdot C.$$

Proof. Let $(C_t)_{t \in T}$ be a deformation family of C . Since C is integral, a general deformation C_t is integral. Since $K_X \cdot C_t < 0$ for all $t \in T$ we see that the locus

$$S := \overline{\cup_{t \in T} C_t}$$

has the property that $K_X|_S$ is not pseudoeffective. Since T and the general C_t are irreducible, the graph of the family $\mathcal{C} \rightarrow T$ is irreducible. Thus S is irreducible, and since K_X is pseudoeffective, it is a surface. By Lemma 4.1 there exists a $j \in \{1, \dots, r\}$ such that $S = S_j$.

In order to prove the uniqueness of the surface S_j and the inequality (7) it is sufficient to show that $S_j \cdot C < 0$. The deformation family $(C_t)_{t \in T}$ has no fixed component, in particular for every $k \in \{1, \dots, r\}$ such that $k \neq j$ there exists a $t \in T$ general such that $C_t \not\subset S_k$. Thus we have $S_k \cdot C = S_k \cdot C_t \geq 0$

for every $k \neq j$. Moreover the restriction $N(K_X)|_{S_j}$ is pseudoeffective and the family $(C_t)_{t \in T}$ covers S_j , so

$$N(K_X) \cdot C = N(K_X)|_{S_j} \cdot C = N(K_X)|_{S_j} \cdot C_t \geq 0.$$

Since

$$0 > K_X \cdot C = \sum_{j=1}^r \lambda_j S_j \cdot C + N(K_X) \cdot C,$$

this implies that $S_j \cdot C < 0$. This shows also that *every* deformation family of C lies in S_j . Thus the surface S_j does not depend on the choice of T . \square

5.5. Lemma. *Let \hat{S} be a smooth projective surface that is uniruled, and let $\hat{C} \subset \hat{S}$ be an irreducible curve.*

a) Suppose that $K_{\hat{S}} \cdot \hat{C} < 0$. Then there exists an effective 1-cycle $\sum \alpha_k C_k$ with coefficients in \mathbb{Q}^+ such that

$$[\sum \alpha_k C_k] = [\hat{C}]$$

and C_1 is a rational curve such that $K_{\hat{S}} \cdot C_1 < 0$.

b) Suppose that $K_{\hat{S}} \cdot \hat{C} \leq -4$. Then there exists an effective 1-cycle $\sum_{k=1}^m C_k$ with $m \geq 2$ such that

$$[\sum_{k=1}^m C_k] = [\hat{C}]$$

such that $K_{\hat{S}} \cdot C_1 < 0$ and $K_{\hat{S}} \cdot C_2 < 0$.

Proof. The statements are trivial for $\hat{S} \simeq \mathbb{P}^2$, so suppose that this is not the case. Fix an ample line bundle H on \hat{S} . We will argue by induction on the degree $d := H \cdot \hat{C}$. We start the induction with $d = 0$ where both statements are trivial. Let us now do the induction step.

Proof of a) If \hat{C} is a rational curve we are finished. If \hat{C} is not rational we know by Theorem 5.1 that $\dim_{\hat{C}} \text{Chow}(S) \geq 1$, so we have a positive-dimensional family of deformations $(C_t)_{t \in T}$.

1st case. The family of deformations splits. Thus there exists a cycle $\sum \beta_l C_l$ that is not integral such that $[\sum \beta_l C_l] = [\hat{C}]$. Up to renumbering we can suppose that $K_{\hat{S}} \cdot C_1 < 0$. Since $H \cdot C_1 < H \cdot \hat{C}$ we can apply the induction hypothesis to C_1 and conclude.

2nd case. The family of deformations does not split. We claim that in this case \hat{S} is a ruled surface or \mathbb{P}^2 . Arguing by contradiction, let $\sigma : \hat{S} \rightarrow S_0$ be a MMP to some Mori fibre space S_0 which we can suppose to be a ruled surface⁶. Let E be the σ -exceptional locus, then $\hat{C} \cdot E > 0$ since the deformations of \hat{C} cover \hat{S} and do not split. Thus the family $(\sigma(C_t))_{t \in T}$ is positive-dimensional, does not split and has a fixed point $p \in \sigma(E)$. This contradicts [Pet01, Lemma 3.3].

⁶If S_0 is \mathbb{P}^2 just take the same MMP but omit the last blow-up.

Since by our assumption $\hat{S} \not\cong \mathbb{P}^2$, this shows that \hat{S} is a ruled surface. We claim that $[\hat{C}]$ lies in the interior of the Mori cone $\overline{\text{NE}}(\hat{S})$: otherwise $\overline{\text{NE}}(\hat{S})$ would be generated by $[\hat{C}]$ and $[F]$ where F is a fibre of the ruling⁷. Since $K_{\hat{S}} \cdot \hat{C} < 0$ we obtain that \hat{S} is a del Pezzo surface, hence \mathbb{F}_1 or $\mathbb{P}^1 \times \mathbb{P}^1$. In both cases every curve in the extremal ray $\mathbb{R}^+[\hat{C}]$ is rational, a contradiction. Thus we can choose a curve $B \subset \hat{S}$ such that $[C]$ lies in the interior of the cone generated by $[B]$ and $[F]$, where F is a fibre of the ruling. Thus we have

$$[C] = \lambda_1[F] + \lambda_2[B],$$

with $\lambda_i \in \mathbb{Q}^+$.

Proof of b). By Theorem 5.1 we have $\dim_{\hat{C}} \text{Chow}(S) \geq 2$, so we have a two-dimensional family of deformations $(C_t)_{t \in T}$. Note that the family of deformations splits: arguing by contradiction, let $\sigma : \hat{S} \rightarrow S_0$ be a MMP to some Mori fibre space S_0 which we can suppose to be a ruled surface. Since T has dimension at least two, the subfamily T_p parametrising the deformations through a general point $p \in \hat{S}$ has dimension at least one. Thus $(\sigma(C_t))_{t \in T_p}$ is positive-dimensional, does not split and has a fixed point $p \in S_0$. This contradicts [Pet01, Lemma 3.3].

Since the deformations split, there exists a cycle $\sum_{l=1}^{m'} C_l$ with $m' \geq 2$ such that $[\sum C_l] = [\hat{C}]$. Up to renumbering we can suppose

$$K_{\hat{S}} \cdot C_1 \leq K_{\hat{S}} \cdot C_2 \leq \dots \leq K_{\hat{S}} \cdot C_{m'}.$$

If $K_{\hat{S}} \cdot C_2 < 0$ we are finished. If not we have $K_{\hat{S}} \cdot C_1 \leq K_{\hat{S}} \cdot \hat{C}$. Since $H \cdot C_1 < H \cdot \hat{C}$ we can apply the induction hypothesis to C_1 to conclude. \square

5.6. Lemma. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let S_1, \dots, S_r be the surfaces appearing in the divisorial Zariski decomposition (5). Set*

$$b := \max\{1, -K_X \cdot Z \mid Z \text{ a curve s.t. } Z \subset S_{j,\text{sing}} \text{ or } Z \subset S_j \cap S_{j'}\}.$$

If $C \subset X$ is a curve such that

$$-K_X \cdot C > b,$$

then we have $\dim_C \text{Chow}(X) > 0$.

Proof. Since $b \geq 1$ the curve C is not very rigid by Theorem 4.5. Let $m \in \mathbb{N}$ be minimal such that $\dim_{mC} \text{Chow}(X) > 0$, and let $(C_t)_{t \in T}$ be a family of deformations. Let $\mathcal{C} \rightarrow T$ be the graph of the family: up to replacing \mathcal{C} by an irreducible component $\mathcal{C}' \subset \mathcal{C}$ that contains C we can suppose that \mathcal{C} is irreducible. In particular the locus covered by the effective 1-cycles C_t is irreducible. The family $(C_t)_{t \in T}$ has no fixed component, i.e. there does not exist a curve B that is contained in the support of every C_t . Indeed since mC is a member of the family, we would have $B = C$, but then

⁷Note that $[\hat{C}] \notin \mathbb{R}^+[F]$ since \hat{C} is not rational

$\dim_{(m-1)C} \text{Chow}(X) > 0$, contradicting the minimality of m . Thus if C_t is general and

$$C_t = \sum_l \alpha_l C_{t,l}$$

its decomposition, we have $\dim_{C_{t,l}} \text{Chow}(X) > 0$ for all l . Up to renumbering we can suppose that $K_X \cdot C_{t,1} < 0$. By Lemma 5.4 applied to $C_{t,1}$ there exists a unique surface S_j from the decomposition (5) such that

$$\overline{\bigcup_{t \in T \text{ general}} C_{t,1}} = S_j.$$

Since the locus covered by the family $(C_t)_{t \in T}$ is irreducible, we see that $C \subset S_j$.

By the definition of b we have $C \not\subset S_l$ for every $l \neq j$, thus we have

$$S_l \cdot C \geq 0 \quad \forall l \neq j.$$

The restriction $N(K_X)|_{S_j}$ is pseudoeffective and for $t \in T$ general all the curves $C_{t,l}$ are movable in S_j , so we get

$$N(K_X) \cdot C = N(K_X)|_{S_j} \cdot C = N(K_X)|_{S_j} \cdot C_t = \sum_l \alpha_l (N(K_X)|_{S_j} \cdot C_{t,l}) \geq 0.$$

Since $K_X \cdot C < 0$ the equality (5) now implies $S_j \cdot C < 0$. Thus we have

$$K_{S_j} \cdot C < K_X \cdot C < -b.$$

By definition of b the curve C is not contained in the singular locus of S_j . Let $\pi_j : \hat{S}_j \rightarrow S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 4.A). The strict transform \hat{C} of C is well-defined and by (4) we have

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < -b.$$

Since $b \geq 1$ we obtain by Theorem 5.1 that

$$\dim_{\hat{C}} \text{Chow}(\hat{S}) > 0,$$

so \hat{C} deforms. Thus its push-forward $\pi_* \hat{C} = C$ deforms. \square

5.7. Corollary. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let b be the constant from Lemma 5.6 and set*

$$d := \max\{3, b\}.$$

If $C \subset X$ is a curve such that $-K_X \cdot C > d$, then we have

$$[C] = [C_1] + [C_2]$$

with C_1 and C_2 effective 1-cycles (with integer coefficients) on X .

Proof. Since $d \geq b$ we know by Lemma 5.6 that $\dim_C \text{Chow}(X) > 0$. Let $(C_t)_{t \in T}$ be a family of deformations. By Lemma 5.4 there exists a unique surface S_j from the divisorial Zariski decomposition (5) such that the C_t are contained in the surface S_j . Moreover we have

$$K_{S_j} \cdot C < K_X \cdot C < d.$$

By the definition of the constant b in Lemma 5.6 we have $C \not\subset S_{j,\text{sing}}$.

Let $\pi_j : \hat{S}_j \rightarrow S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 4.A). The strict transform \hat{C} of C is well-defined and by (4) we have

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < -3.$$

Thus by Lemma 5.5,b) there exists an effective 1-cycle $\sum_{k=1}^m C_k$ with $m \geq 2$ such that

$$[\sum_{k=1}^m C_k] = [\hat{C}]$$

such that $K_{\hat{S}_j} \cdot C_1 < 0$ and $K_{\hat{S}_j} \cdot C_2 < 0$. Since $K_{\hat{S}_j}$ is π_j -nef, we have $(\pi_j)_* C_1 \neq 0$ and $(\pi_j)_* C_2 \neq 0$. Thus we obtain

$$[C] = (\pi_j)_* [\hat{C}] = \sum_{k=1}^m (\pi_j)_* [C_k]$$

and the first two terms of this sum are not zero. \square

5.8. Lemma. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X -negative extremal ray in $\overline{NE}(X)$ where Γ_i is a curve that is not very rigid. Then the following holds:*

- a) *There exists a curve $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$ and $\dim_C \text{Chow}(X) > 0$.*
- b) *There exists a rational curve $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$.*

Proof. Let

$$m := \min\{k \in \mathbb{N} \mid \exists C \subset X \text{ curve s.t. } [C] \in \mathbb{R}^+[\Gamma_i], \dim_{kC} \text{Chow}(X) > 0\}.$$

Our goal is to show that $m = 1$. Arguing by contradiction let C be a curve that realises the minimal degree $m > 1$. We have $\dim_{mC} \text{Chow}(X) > 0$, so let $(C_t)_{t \in T}$ be a family of deformations. Note that this family has no fixed component, since we chose m to be minimal. Thus if C_t is a general member of the family, then for every irreducible component $C'_t \subset C_t$ we have $\dim_{C'_t} \text{Chow}(X) > 0$. Since the ray $\mathbb{R}^+[\Gamma_i]$ is extremal, we have $C'_t \in \mathbb{R}^+[\Gamma_i]$, a contradiction.

For the proof of statement b), choose $C \in \mathbb{R}^+[\Gamma_i]$ such that $\dim_C \text{Chow}(X) > 0$. By Lemma 5.4 the deformations of C are contained in one of the surfaces S_j from the divisorial Zariski decomposition (5) and we have

$$K_{S_j} \cdot C < K_X \cdot C < 0.$$

Up to replacing C by a general deformation we can suppose that $C \not\subset S_{j,\text{sing}}$. Let $\pi_j : \hat{S}_j \rightarrow S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 4.A). The strict transform \hat{C} of C is well-defined and by (4) we have

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < 0.$$

By Lemma 5.5,a) there exists an effective 1-cycle $\sum \alpha_k C_k$ with coefficients in \mathbb{Q}^+ such that

$$[\sum \alpha_k C_k] = [\hat{C}]$$

and C_1 is a rational curve such that $K_{\hat{S}_j} \cdot C_1 < 0$. Since $K_{\hat{S}_j}$ is π_j -nef, we have $(\pi_j)_* C_1 \neq 0$. Thus we obtain

$$[C] = (\pi_j)_* [\hat{C}] = \sum_k \alpha_k (\pi_j)_* [C_k],$$

and the first term $\alpha_1 (\pi_j)_* [C_1]$ is not zero. Since the ray $\mathbb{R}^+ [\Gamma_i]$ is extremal, we see that $[C_1]$ is a positive multiple of $[\Gamma_i]$. \square

6. CONE THEOREMS

6.1. Lemma. *Let X be a normal compact Kähler threefold such that K_X is \mathbb{Q} -Cartier. Let $N \subset N_1(X)$ be a closed convex cone that contains no lines. Suppose that there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i \in I}$ of curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$N = \overline{N_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]},$$

Then we have

$$N = N_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].$$

Our argument follows the proof of the cone theorem for projective manifolds in [Deb01].

Proof. Set $V := N_{K_X \geq 0} + \sum_i \mathbb{R}^+ [\Gamma_i]$. We want to prove that $\overline{V} = V$, i.e. the cone V is closed. Since V contains no lines, it is the convex hull of its extremal rays [Deb01, Lemma 6.7 b)]. It is thus sufficient to show that if $\mathbb{R}^+ [r]$ is an extremal ray in \overline{V} such that $K_X \cdot r < 0$, then $\mathbb{R}^+ [r]$ is in V .

Let ω be a Kähler form on X and choose a $\varepsilon > 0$ such that

$$(K_X + \varepsilon \omega) \cdot r < 0.$$

Then we have by our hypothesis

$$N = \overline{N_{K_X + \varepsilon \omega \geq 0} + \sum_{j \in J} \mathbb{R}^+ [\Gamma_j]}$$

where the sum runs over those indices $j \in I$ such that $(K_X + \varepsilon\omega) \cdot \Gamma_j < 0$. Since $-K_X \cdot \Gamma_j \leq d$ we have $\omega \cdot \Gamma_j \leq \frac{d}{\varepsilon}$. Since the degree of the cohomology classes $[\Gamma_j]$ with respect to ω is bounded, there are only finitely many such classes. In particular we have

$$N = N_{K_X + \varepsilon\omega \geq 0} + \sum_{j \in J} \mathbb{R}^+[\Gamma_j]$$

and up to renumbering we can suppose that $J = \{1, \dots, q\}$ for some $q \in \mathbb{N}$. Write now r as the limit of a sequence $r_m + s_m$ such that for all $m \in \mathbb{N}$ one has

$$(K_X + \varepsilon\omega) \cdot r_m \geq 0$$

and $s_m \in \sum_{j=1}^q \mathbb{R}^+[\Gamma_j]$, that is

$$s_m = \sum_{j=1}^q \lambda_{j,m} \Gamma_j.$$

Since $\omega \cdot (r_m + s_m)$ converges to $\omega \cdot r$ the sequences $\omega \cdot r_m$ and $\omega \cdot \lambda_{j,m} \Gamma_j$ are bounded by $\omega \cdot r + 1$ for large m . In particular we can assume after taking subsequences that the sequences r_m and $\lambda_{j,m} \Gamma_j$ converge. Since

$$r = \lim_{m \rightarrow \infty} r_m + \sum_{j=1}^q \lim_{m \rightarrow \infty} \lambda_{j,m} \Gamma_j.$$

and r is extremal in \overline{V} this implies that $\lim_{m \rightarrow \infty} r_m$ and $\lim_{m \rightarrow \infty} \lambda_{j,m} \Gamma_j$ are non-negative multiples of r . Since $(K_X + \varepsilon\omega) \cdot \lim_{m \rightarrow \infty} r_m \geq 0$ and $(K_X + \varepsilon\omega) \cdot r < 0$ we get that $\lim_{m \rightarrow \infty} r_m = 0$. This implies that there exists a $j_0 \in \{1, \dots, q\}$ such that $\lim_{m \rightarrow \infty} \lambda_{j_0,m} z_{j_0}$ is a positive multiple of r . Hence the extremal rays $\mathbb{R}^+ r$ and $\mathbb{R}^+ z_{j_0}$ coincide, so $\mathbb{R}^+ r$ is in V . \square

6.2. Theorem. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i \in I}$ of curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$$

If the ray $\mathbb{R}^+[\Gamma_i]$ is extremal in $\overline{NE}(X)$, there exists a rational curve C_i on X such that $[C_i] \in \mathbb{R}^+[\Gamma_i]$.

Proof of Theorem 6.2. Let $d \in \mathbb{N}$ be the bound from Corollary 5.7. There are only countably many curve classes $[C] \subset \overline{NE}(X)$ and we choose a representative Γ_i for each class such that $0 < -K_X \cdot \Gamma_i \leq d$. We set

$$V := \overline{NE}(X)_{K_X \geq 0} + \sum_{0 < -K_X \cdot \Gamma_i \leq d} \mathbb{R}^+[\Gamma_i].$$

Fix a Kähler form ω on X such that

$$\omega \cdot C \geq 1$$

for every curve $C \subset X$ ⁸.

Step 1. We have $\overline{NE}(X) = V$. By Lemma 6.1 it is sufficient to prove that $\overline{NE}(X) = \overline{V}$, i.e. the class $[C]$ of every irreducible curve $C \subset X$ is contained in V . We will prove the statement by induction on the degree $l := \omega \cdot C$. The start of the induction for $l = 0$ is trivial. Suppose now that we have shown the statement for all curves of degree at most $l - 1$ and let C be a curve such that $l - 1 < \omega \cdot C \leq l$. If $-K_X \cdot C \leq d$ we are done. Otherwise there exists by Corollary 5.7 a decomposition

$$[C] = [C_1] + [C_2]$$

with C_1 and C_2 effective 1-cycles (with integer coefficients) on X . Since $\omega \cdot C_i \geq 1$ for $i = 1, 2$ we have $\omega \cdot C_i \leq l - 1$ for $i = 1, 2$. By induction both classes are in V , so $[C]$ is in V .

Step 2. Every extremal ray contains the class of a rational curve. If the ray $\mathbb{R}^+[\Gamma_i]$ is extremal in $\overline{NE}(X)$ we know by Theorem 4.5 and Lemma 5.8 that there exists a rational curve C_i such that $[C_i]$ is in the extremal ray. \square

6.3. Theorem. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i \in I}$ of curves on X such that*

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$\overline{NA}(X) = \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$$

If the ray $\mathbb{R}^+[\Gamma_i]$ is extremal in $\overline{NA}(X)$, there exists a rational curve C_i on X such that $[C_i] \in \mathbb{R}^+[\Gamma_i]$.

Theorem 6.3 is a consequence of Theorem 6.2 and the following proposition.

6.4. Proposition. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective.*

Suppose that there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i \in I}$ of curves on X such that

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$$

⁸ The form ω exists since the cohomology classes of curves in X form a discrete set in $\overline{NE}(X)$, so for every Kähler form ω' there exists a real constant $\lambda > 0$ such that

$$\omega' \cdot C \geq \lambda$$

for every curve $C \subset X$.

Then we have

$$\overline{NA}(X) = \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]$$

Proof. Set

$$V := \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].$$

By Lemma 6.1 it is sufficient to show that $\overline{NA}(X) \subset \overline{V}$. Let $\pi : \hat{X} \rightarrow X$ be a desingularisation and consider $\alpha \in \overline{NA}(X)$. By Lemma 3.10, there exists $\hat{\alpha} \in \overline{NA}(\hat{X})$ such that $\alpha = \pi_*(\hat{\alpha})$. By [DP04, Cor.0.3] the cone $\overline{NA}(\hat{X})$ is the closure of the convex cone generated by cohomology classes of the form $[\hat{\omega}]^2$, $[\hat{\omega}] \cdot [\hat{S}]$ and $[\hat{C}]$ where $\hat{\omega}$ is a Kähler form, \hat{S} a surface and \hat{C} a curve on \hat{X} . Our goal is to show that the π -image of any of this three types is contained in \overline{V} .

1st case. $\hat{\alpha} = [\hat{\omega}]^2$ with $\hat{\omega}$ a Kähler form. Since $\pi^*(K_X)$ is pseudoeffective, we have $\pi^*(K_X) \cdot [\hat{\omega}]^2 \geq 0$, hence

$$K_X \cdot \alpha = K_X \cdot \pi_*(\hat{\alpha}) = \pi^*(K_X) \cdot \hat{\alpha} \geq 0,$$

and thus $\alpha \in \overline{NA}(X)_{K_X \geq 0}$.

2nd case. $\hat{\alpha} = [\hat{C}]$ with \hat{C} a curve. Then set $C := \pi_*(\hat{C})$, so that $\alpha = [C]$. Since we have an inclusion

$$(8) \quad \overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i] \hookrightarrow \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i],$$

and by hypothesis any curve class $[C]$ is in the left hand side, we see that $[C] \in V$.

(3) 3rd case. $\hat{\alpha} = [\hat{\omega}] \cdot [\hat{S}]$ with \hat{S} an irreducible surface and $\hat{\omega}$ a Kähler form. If

$$\pi^*(K_X) \cdot [\hat{\omega}] \cdot [\hat{S}] \geq 0,$$

the class α is in $\overline{NA}(X)_{K_X \geq 0}$. Suppose now that

$$\pi^*(K_X) \cdot [\hat{\omega}] \cdot [\hat{S}] < 0.$$

We claim that α is in the image of the inclusion (8) which concludes the proof.

Proof of the claim. Using the projection formula we see that $\pi(\hat{S})$ is not a point. Since X has at most isolated singularities we see that $\pi(\hat{S})$ is not a curve, so we can suppose that $\pi(\hat{S})$ is a surface S . Since $\pi^*(K_X) \cdot [\omega] \cdot [S] < 0$, the restriction $\pi^*(K_X)|_{\hat{S}}$ is not pseudoeffective. Thus the restriction $K_X|_S$ is not pseudo-effective. By Lemma 4.1 the surface S is one of the surfaces S_j from the Zariski decomposition (5), moreover it is projective and uniruled. Let $\pi_j : \hat{S}_j \rightarrow S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 4.A). Since \hat{S}_j is uniruled, we have $H^2(\hat{S}_j, \mathcal{O}_{\hat{S}_j}) = 0$. In particular the Chern class map

$$\text{Pic } \hat{S}_j \rightarrow H^2(\hat{S}_j, \mathbb{Z})$$

is surjective, so

$$NS_{\mathbb{R}}(\hat{S}_j) = H^2(\hat{S}_j, \mathbb{R}) \cap H^{1,1}(\hat{S}_j).$$

Thus the pull-back $\pi_j^* \omega$ which is a real closed form of type $(1, 1)$ is represented by a \mathbb{R} -divisor which is nef and big. In particular it is a limit of classes of ample \mathbb{Q} -divisors H_m on S which we can represent by classes of effective 1-cycles $[C_m]$ with rational coefficients. We claim that the sequence

$$\pi_*[C_m] \in \overline{\text{NE}}(X) \subset N_1(X)$$

converges to $[\omega] \cdot [S]$. By duality it is sufficient to prove that for every η a real closed $(1, 1)$ -form on X , the sequence $[\eta] \cdot \pi_*[C_m]$ converges to $[\eta] \cdot [\omega] \cdot [S]$. Yet by the projection formula we have

$$[\eta] \cdot \pi_*[C_m] = \pi^*[\eta] \cdot [C_m]$$

and

$$[\eta] \cdot [\omega] \cdot [S] = [\eta|_S] \cdot [\omega|_S] = \pi^*[\eta] \cdot \pi^*[\omega],$$

so the convergence follows from the construction of the sequence $[C_m]$. \square

Proof of Theorem 1.2. The only statement that is not part of Theorem 6.3 is that in every K_X -negative extremal ray R_i we can find a rational curve Γ_i such that $\Gamma_i \in R_i$ and

$$0 < -K_X \cdot \Gamma_i \leq 4.$$

If the extremal ray is small, this is clear by Theorem 4.5. If the extremal ray is divisorial, the contraction exists by Theorem 1.3⁹. Thus we can conclude by [Deb01, Thm.7.46]. \square

6.5. Remark. The cone theorem 1.2 actually holds if X has at most canonical singularities: using a relative MMP (which exists since we can take a resolution of singularities which is a projective morphism) we construct a bimeromorphic morphism $\mu : X' \rightarrow X$ such that $K'_X \simeq \mu^* K_X$ and X' has at most terminal singularities. Since $\mu_*(\overline{\text{NA}}(X')) = \overline{\text{NA}}(X)$ by Proposition 3.10, the statement follows from our theorem.

7. CONTRACTIONS OF EXTREMAL RAYS

For the whole section we make the following

Assumption. Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. We fix $R := \mathbb{R}^+[\Gamma_{i_0}]$ a K_X -negative extremal ray in $\overline{\text{NA}}(X)$.

7.1. Definition. We say that the K_X -negative extremal ray R is small if every curve $C \subset X$ with $[C] \in R$ is very rigid in the sense of Definition 4.3. Otherwise we say that the extremal ray R is divisorial.

⁹The proof of Theorem 1.3 uses only Theorem 6.3.

7.2. Remark. If the extremal ray R admits a Mori contraction $\varphi : X \rightarrow Y$, the contraction is small (resp. divisorial) if and only if this holds for the extremal ray. Indeed if the extremal ray is small, then by Theorem 4.5 we have $-K_X \cdot C \leq 1$ for every curve such that $[C] \in R$. In particular there are only finitely many cohomology classes in R that are represented by curves. Since for every class there are only finitely many deformation families and by hypothesis every curve in the ray is very rigid, we see that there are only finitely many curves $C \subset X$ such that $[C] \in R$. Thus φ contracts only finitely many curves, it is small.

7.3. Proposition. *There exists a nef class $\alpha \in N^1(X)$ such that*

$$R = \{z \in \overline{NA}(X) \mid \alpha \cdot z = 0\},$$

such that, using the notation of Theorem 6.3, the class α is strictly positive on

$$\left(\overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\}.$$

We call α a nef supporting class for the extremal ray R .

Proof. We set

$$V := \overline{NA}(X)_{K_X \geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i].$$

By Lemma 6.1 the cone V is closed and by Theorem 6.3 we have

$$\overline{NA}(X) = V + \mathbb{R}^+[\Gamma_{i_0}]$$

By [Deb01, Lemma 6.7(d)] there exists a linear form on $N_1(X)$ which vanishes on R and is positive on $V \setminus \{0\}$. This linear form gives the class α by Proposition 3.7. \square

7.A. Divisorial rays. The following proposition is a particular case of [AT84, Thm.2] and a generalisation of the well-known theorem of Grauert [Gra62].

7.4. Proposition. *Let X be a normal compact complex space, and let S be a prime divisor that is \mathbb{Q} -Cartier of Cartier index m . Suppose that S admits a morphism with connected fibres $f : S \rightarrow B$ such that $\mathcal{O}_S(-mS)$ is f -ample. Then there exists a bimeromorphic morphism $\varphi : X \rightarrow Y$ to a normal compact complex space Y such that $\varphi|_S = f$ and $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus B$.*

Proof. Since $\mathcal{O}_S(-mS)$ is f -ample there exists a multiple m' of m such that we have

$$R^1 f_*(\mathcal{O}_S(-km'S)) = 0$$

for all $k \in \mathbb{N}$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf corresponding to the Cartier divisor $-m'D$, then $\mathcal{I}/\mathcal{I}^2|_S$ is an f -ample line bundle on S . Since we have

$$R^1 f_*(\mathcal{I}^k/\mathcal{I}^{k+1}) = R^1 f_*(\mathcal{O}_S(-km'S)) = 0$$

for all $k \in \mathbb{N}$, the natural morphisms

$$f_*(\mathcal{O}_X/\mathcal{I}^{k+1}) \rightarrow f_*(\mathcal{O}_X/\mathcal{I}^k)$$

are surjective for all $k \in \mathbb{N}$. Thus we can apply [AT84, Thm.2] to conclude. \square

7.5. Lemma. *Suppose that the extremal ray R is divisorial. We set*

$$S = \bigcup_{C \subset X, [C] \in R} C.$$

Then S is an irreducible projective surface and $S \cdot C < 0$ for all curves C with $[C] \in R$.

Proof. By definition the extremal ray R contains a class $[C]$ with C a curve that is not very rigid. By Lemma 5.8,a) we may suppose that $\dim_C \text{Chow}(X) > 0$. By Lemma 5.4 there exists a unique surface S_j from the divisorial Zariski decomposition (5) such that C and its deformations are contained in S_j , moreover we have $S_j \cdot C < 0$.

Let now C' be any curve such that $[C'] \in R$. Then we have $[C'] = \lambda[C]$ for some $\lambda \in \mathbb{Q}^+$, hence

$$S_j \cdot C' = \lambda S_j \cdot C < 0.$$

Thus we have $C' \subset S_j$ and $S = S_j$. The projectivity of S is shown in Lemma 4.1. \square

7.6. Notation. Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$ is divisorial, and let S be the surface from Lemma 7.5. Let $\nu : \tilde{S} \rightarrow S \subset X$ be the normalisation; then $\nu^*(\alpha)$ is a nef class on \tilde{S} and we may consider the nef reduction

$$\tilde{f} : \tilde{S} \rightarrow \tilde{B}$$

with respect to $\nu^*(\alpha)$, cf. [BCE⁺02, Thm.2.6]. By Lemma 5.8,a) there exists a curve $C \subset X$ such that $[C] \in R$ and $\dim_C \text{Chow}(X) > 0$. Since $S \cdot C < 0$ by Lemma 7.5, the deformations $(C_t)_{t \in T}$ of C induce a dominating family $(\tilde{C}_t)_{t \in T'}$ of \tilde{S} such that $\nu^*(\alpha) \cdot \tilde{C}_t = 0$. By definition of the nef reduction this implies

$$n(\alpha) := \dim \tilde{B} \in \{0, 1\}.$$

7.7. Corollary. *Suppose that the extremal ray R is divisorial and $n(\alpha) = 0$. Then the surface S can be blown down to a point: there exists a birational morphism $\varphi : X \rightarrow Y$ to a normal compact threefold Y with $\dim \varphi(S) = 0$ such that $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus 0$.*

Proof. Since $n(\alpha) = 0$, the class $\nu^*(\alpha)$ is trivial on \tilde{S} [BCE⁺02, Thm.2.6]. Hence $\nu^*(\alpha) \cdot \tilde{C} = 0$ for all curves $\tilde{C} \subset \tilde{S}$, hence $\alpha \cdot C = 0$ for all curves $C \subset S$. By Proposition 7.3 the class of every curve $C \subset S$ belongs to R .

Let m be the Cartier index of S . The morphism

$$\overline{\text{NE}}(S) \rightarrow \overline{\text{NE}}(X)$$

has image in the ray R , so the divisor $\mathcal{O}_S(-mS)$ is strictly positive on $\overline{\text{NE}}(S) \setminus 0$. Since S is projective by Lemma 7.5 we see by [Laz04, Thm.1.4.29] that $\mathcal{O}_S(-mS)$ is ample and conclude with Proposition 7.4. \square

7.8. Lemma. *Suppose that the extremal ray R is divisorial and $n(\alpha) = 1$. Then there exists a fibration $f : S \rightarrow B$ onto a curve B such that a curve $C \subset S$ is contracted if and only if $[C] \in R$.*

Proof. Let \tilde{C} be a general fibre of the nef reduction $\tilde{f} : \tilde{S} \rightarrow \tilde{B}$ with respect to $\nu^*(\alpha)$, then we have $\nu^*(\alpha) \cdot \tilde{C} = 0$. By definition of α this implies that for $C := \nu(\tilde{C})$ we have $[C] \in R$. Thus we have $K_X \cdot C < 0$ and the cycle space $\text{Chow}(X)$ has positive dimension in C , so by Lemma 5.4 we have

$$K_S \cdot C < K_X \cdot C < 0.$$

The normalisation ν being finite and X having only finitely many singular points, we can choose \tilde{C} such that the following three conditions hold:

- $C \subset X_{\text{nons}}$,
- $C \not\subset S_{\text{sing}}$, and
- $\tilde{C} \subset \tilde{S}_{\text{nons}}$,

By the first property the intersection numbers $K_X \cdot C < 0$ and $S \cdot C < 0$ are integers, hence $K_S \cdot C \leq -2$. By the second property and (4) we have

$$K_{\tilde{S}} \cdot \tilde{C} \leq K_S \cdot C \leq -2$$

with equality if and only if $C \subset S_{\text{nons}}$ and $K_X \cdot C = -1$ and $S \cdot C = -1$. Since \tilde{C} is a fibre of the fibration \tilde{f} contained in the smooth locus of \tilde{S} , we have

$$-2 \leq \deg K_{\tilde{C}} = K_{\tilde{S}} \cdot \tilde{C} \leq -2.$$

Thus $\tilde{C} = C$ is a rational curve contained in S_{nons} such that $K_S \cdot C = -2$. In particular the sheaf $\mathcal{O}_S(C)$ is invertible. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(mC) \rightarrow \nu_* \mathcal{O}_{\tilde{S}}(mC) \rightarrow \mathcal{F} \otimes \mathcal{O}_{\tilde{S}}(mC) \rightarrow 0,$$

where \mathcal{F} is a coherent sheaf supported on the non-normal locus of S . Since $C \subset S_{\text{nons}}$, we have $\mathcal{F} \otimes \mathcal{O}_{\tilde{S}}(mC) \simeq \mathcal{F}$ for all $m \in \mathbb{N}$. Since $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(mC))$ goes to infinity as $m \rightarrow \infty$ we see that $\kappa(S, \mathcal{O}_S(C)) = 1$. Since $C^2 = 0$ we see that some multiple of C defines a fibration $f : S \rightarrow B$ onto a curve B such that we have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\nu} & S \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{B} & \longrightarrow & B \end{array}$$

\square

7.9. Corollary. *Suppose that the extremal ray R is divisorial and $n(\alpha) = 1$. Let $f : S \rightarrow B$ be the fibration defined in Lemma 7.8. Then the surface S can be contracted onto a curve: there exists a bimeromorphic morphism $\varphi : X \rightarrow Y$ to a normal compact threefold Y such that $\varphi|_S = f$ and $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus B$.*

Proof. Let m be the Cartier index of S . By Lemma 7.8 the surface S admits a fibration f onto a curve contracting exactly the curves in R . By Lemma 7.5 the divisor $\mathcal{O}_X(-mS)$ is ample on every curve in R , so its restriction $\mathcal{O}_S(-mS)$ is f -ample. Conclude with Proposition 7.4. \square

7.B. Small rays.

7.10. Notation. Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$ is small. Set

$$C := \cup_{C_l \subset X, [C_l] \in R} C_l,$$

then C is a finite union of curves by Remark 7.2. We say that C is contractible if there exists a bimeromorphic morphism $\varphi : X \rightarrow Y$ onto a normal threefold Y with $\dim \varphi(C) = 0$ such that $\varphi|_{X \setminus C}$ is an isomorphism onto $Y \setminus \varphi(C)$.

7.11. Lemma. *Let X be a normal threefold, and let $C = \cup_{j=1}^r C_l \subset X$ be a finite union of irreducible compact curves. Let S be an effective irreducible Weil divisor which is \mathbb{Q} -Cartier such that $S \cdot C_l < 0$ for all $j \in \{1, \dots, r\}$. Let m be a positive integer such that mS is Cartier. Let H be an effective Cartier divisor on mS such that $H \cdot C_l < 0$ for all $j \in \{1, \dots, r\}$. Then C is contractible.*

The argument is inspired by [Kol89, Lemma 4.10].

Proof. The divisor H is a 1-dimensional complex subspace of X that is a locally complete intersection, so its conormal sheaf $N_{H/X}^*$ is a locally free sheaf of rank 2 and an extension

$$0 \rightarrow N_{H/mS}^* \simeq \mathcal{O}_{mS}(-H)|_H \rightarrow N_{H/X}^* \rightarrow N_{mS/X}^*|_H \simeq \mathcal{O}_X(-mS)|_H \rightarrow 0.$$

By our hypothesis on S and H the restriction of $N_{H/X}^*$ to any of the curves C_l is an extension of ample line bundles, so the restriction $N_{H/X}^*|_C$ is ample. Let $\mathcal{I} \subset \mathcal{O}_X$ be the biggest ideal sheaf that agrees with $\mathcal{I}_H \subset \mathcal{O}_X$ in the generic point of every irreducible component $C_l \subset C$ ¹⁰. Then the support of $\mathcal{I}/\mathcal{I}^2$ is C and the natural morphism

$$N_{H/X}^* = \mathcal{I}_H/\mathcal{I}_H^2 \rightarrow \mathcal{I}/\mathcal{I}^2$$

is an isomorphism in the generic point of every curve $C_l \subset C$. Thus $\mathcal{I}/\mathcal{I}^2$ is ample [Anc82], hence C is contractible by [AT84, Cor.3]. \square

¹⁰ The sheaf \mathcal{I} defines the scheme-theoretic image [Har77, II, Ex.3.11(d)] of the natural map $\cup_j C_{j,gen} \rightarrow X$ where we endowed the generic points $C_{j,gen} \subset H$ with the open subscheme structure.

7.12. Proposition. *Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small. Suppose that there exists an integral surface $S \subset X$ such that $S \cdot \Gamma_i < 0$. Then we have $\alpha^2 \cdot S > 0$.*

Proof. By hypothesis the cohomology class $\alpha - (K_X + S)$ is positive on the extremal ray R , moreover we know by Proposition 7.3 that α is positive on

$$\left(\overline{\text{NA}}(X)_{K_X \geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\}.$$

Thus, up to replacing α by some positive multiple, we can suppose that $\alpha - (K_X + S)$ is positive on $\overline{\text{NA}}(X) \setminus \{0\}$. Since X is a Kähler space this implies by Corollary 3.14 that $\alpha - (K_X + S)$ is a Kähler class. In particular we have

$$[\alpha - (K_X + S)]^2 \cdot S > 0.$$

Let $\pi : S' \rightarrow S$ be the composition of normalisation and minimal resolution (cf. Subsection 4.A). By (3) we have $K_{S'} + E = \pi^* K_S$ with E an effective divisor. Since $(\alpha - (K_X + S))|_S$ is a Kähler class, we see that $\pi^* \alpha - (K_{S'} + E)$ is a nef and big class. Note that this implies $\pi^* \alpha \neq 0$: otherwise $-K_{S'} - E$ would be nef and big, so $-K_{S'}$ is big. In particular S' would be projective and contain infinitely many curves on which α is zero, a contradiction.

Since the class $\pi^* \alpha$ is non-zero and nef the Hodge index theorem yields

$$[\pi^* \alpha - (K_{S'} + E)] \cdot \pi^* \alpha > 0.$$

We will now argue by contradiction and suppose that $(\pi^* \alpha)^2 = \alpha^2 \cdot S = 0$. Then the equality above implies

$$(9) \quad -K_{S'} \cdot \pi^* \alpha > E \cdot \pi^* \alpha \geq 0.$$

Thus $K_{S'}$ is not pseudoeffective, hence S' is uniruled and $H^2(S', \mathcal{O}_{S'}) = 0$. In particular we can see the nef class $\pi^* \alpha$ as an \mathbb{R} -divisor on the projective surface S' . Since $\pi^* \alpha$ is nef and S' is a surface, it is an element of $\overline{\text{NM}}(S')$ the closure of the cone of movable curves. The extremal ray R contains only the classes of finitely many curves, so $\pi^* \alpha$ is strictly positive on every movable curve in S' .

Fix an ample divisor A on S' . By [Ara10, Thm.1.3] for every $\varepsilon > 0$ we have a decomposition

$$\pi^* \alpha = C_\varepsilon + \sum \lambda_{i,\varepsilon} M_{i,\varepsilon}$$

where $\lambda_{i,\varepsilon} \geq 0$, $(K_{S'} + \varepsilon A) \cdot C_\varepsilon \geq 0$ and the $M_{i,\varepsilon}$ are movable curves. Since $(\pi^* \alpha)^2 = 0$ and $\pi^* \alpha \cdot M_{i,\varepsilon} > 0$ we see that $\pi^* \alpha = C_\varepsilon$ for all $\varepsilon > 0$. Passing to the limit we obtain $K_{S'} \cdot \pi^* \alpha \geq 0$, a contradiction to (9). \square

7.13. Proposition. *Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small. Suppose that there exists an integral surface $S \subset X$ such that $S \cdot \Gamma_i < 0$. Then C is contractible.*

Proof. By Proposition 7.12 the restriction $\alpha|_S$ is a class that is nef and big. Let $\eta : \tilde{S} \rightarrow S$ be the normalisation, and let $\nu : \hat{S} \rightarrow \tilde{S}$ be the minimal resolution in every point of $\tilde{S}_{\text{sing}} \cap \eta^{-1}(C)$. Set $\pi := \eta \circ \nu$. Then the set-theoretical preimage $\pi^{-1}(C)$ is a finite union of curves that contains the ν -exceptional locus. By Proposition 7.12 the cohomology class $\beta := \pi^* \alpha|_S$ is nef and big. If C' is any effective divisor with support in $\pi^{-1}(C)$ then $\pi_* C'$ has support in C , hence

$$\beta \cdot C' = \alpha \cdot \pi_* C' = 0.$$

By the Hodge index theorem this implies that $(C')^2 < 0$. Thus the intersection matrix of $\pi^{-1}(C)$ is negative definite, hence $\pi^{-1}(C)$ is contractible by Grauert's criterion. By the rigidity lemma the contraction of $\pi^{-1}(C)$ factors through the resolution ν , so we see that $\eta^{-1}(C) \subset \tilde{S}$ is contractible.

Thus there exists a strongly pseudo-convex open neighbourhood $\tilde{U} \subset \tilde{S}$ of $\eta^{-1}(C)$, [Gra62]. If \tilde{U} is chosen small enough, then we have $\eta^{-1}(\eta(\tilde{U})) = \tilde{U}$, since η is a normalisation. Thus $U := \eta(\tilde{U})$ is an open neighborhood of C in S , but it might not be strongly pseudo-convex. We choose a desingularisation $\sigma : X' \rightarrow X$ with strict transform $S' \subset X'$ such that S' is smooth. Note that the map $S' \rightarrow S$ factors via the normalisation η . Let $\tau : S' \rightarrow \tilde{S}$ be the induced map, and set $C' = \tau^{-1}(\tilde{C})$. Then $C' \subset S'$ is contractible and $U' := \tau^{-1}(\tilde{U})$ is a strongly pseudo-convex neighbourhood of C' in S' . Let now m be the Cartier index of $S \subset X$. Then we have

$$\dim H^0(U', \omega_{mU'}|U') = \infty.$$

Let \mathcal{J} be the ideal sheaf of U' in X' . Since U' is strongly pseudo-convex we have

$$\dim H^1(U', \mathcal{J}^k / \mathcal{J}^{k+1} \otimes \omega_{mU'}) < \infty$$

for all $k \in \mathbb{N}$. Thus we have

$$\dim H^0(mU', \omega_{mU'}) = \infty.$$

Via the natural map $(\sigma|_{mU'})_* \omega_{mU'} \hookrightarrow \omega_{mU}$ we see that

$$\dim H^0(mU, \omega_{mU}) \neq 0.$$

By adjunction we have $\omega_{mU} = (\omega_X \otimes \mathcal{O}_X(mS))|_{mU}$. Let l be a positive integer such that lK_X is Cartier. Then $\mathcal{O}_X(lK_X + lmS)|_{mU}$ is a Cartier divisor on mU that is isomorphic to $\omega_{mU}^{\otimes l}$ in the complement of finitely many points. Since the surface U is Cohen-Macaulay (hence S_2), we see that

$$\dim H^0(mU, \mathcal{O}_{mU}(lK_X + lmS)) \neq 0.$$

We conclude by Lemma 7.11. \square

7.14. Theorem. *Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small, and let C be the locus covered by curves in R (cf. Notation 7.10). Then C is contractible.*

Proof. Let $K_X = \sum_{j=1}^r \lambda_j S_j + N(K_X)$ be the divisorial Zariski decomposition (5). Since R is K_X -negative, we have $S_j \cdot R < 0$ for some $j \in \{1, \dots, r\}$ or $N(K_X) \cdot R < 0$. In the first case we conclude by Proposition 7.13. Thus we can suppose that $N(K_X) \cdot R < 0$.

Step 1. Positivity of α . Let $\alpha \in N^1(X)$ be as in Proposition 7.3. By definition of the class α , the class $-N(K_X) + \alpha$ is positive on the extremal ray R . Since α is strictly positive on

$$\left(\overline{\text{NA}}(X)_{K_X \geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+ [\Gamma_i] \right) \setminus \{0\},$$

we can suppose, up to replacing α by some positive multiple, that $-N(K_X) + \alpha$ is strictly positive on this cone. Hence $-N(K_X) + \alpha$ is strictly positive on $\overline{\text{NA}}(X)$. Thus we know by Corollary 3.14 that $-N(K_X) + \alpha$ is a Kähler class, i.e. we have

$$\alpha = N(K_X) + [\omega],$$

where ω is a Kähler form. The class $N(K_X)$ is modified nef, so the class $N(K_X)^2$ is pseudoeffective. Since α is nef we have $\alpha \cdot N(K_X)^2 \geq 0$. Thus we have

$$\alpha^3 = \alpha \cdot N(K_X)^2 + 2\alpha \cdot N(K_X) \cdot [\omega] + \alpha \cdot [\omega]^2 \geq \alpha \cdot [\omega]^2 > 0,$$

where the last inequality follows from the Hodge index theorem. This proves that the class α is big. Hence α contains the class of a Kähler current [DP04, Thm.0.5]. The non-Kähler locus of α contains the union of the curves $C' \subset X$ such that $\alpha \cdot C' = 0$. By the definition of α these are exactly the curves whose class lies in the extremal ray R . However, a priori, the non-Kähler locus may also contain some curves B with $\alpha \cdot B > 0$.

Since α is the sum of a Kähler class and a modified nef class, it is a modified Kähler class [Bou04, Defn.2.2]. Thus by [Bou04, Prop.2.3]¹¹ there exists a modification $\mu : \tilde{X} \rightarrow X$ and a Kähler class $\tilde{\alpha}$ on \tilde{X} such that $\mu_* \tilde{\alpha} = \alpha$. Since $\tilde{\alpha} - \mu^* \alpha$ is μ -nef, the negativity lemma [KM98, Lemma 3.39] implies that we have

$$(10) \quad \mu^* \alpha = \tilde{\alpha} + E$$

with E an effective μ -exceptional \mathbb{R} -divisor. Let now $C_l \subset X$ be an irreducible curve such that $[C_l] \in R$, and set D_l for the support of $\mu^{-1}(C_l)$. Then we have $\alpha|_{C_l} \equiv 0$, so by (10) we have

$$-E|_{D_l} = \tilde{\alpha}|_{D_l}.$$

¹¹Proposition 2.3. in [Bou04] is for compact complex manifolds, but the proof goes through without changes for a variety with isolated singularities. A different way to obtain the decomposition is to take a resolution of singularities $\nu : X' \rightarrow X$ and consider the nef and big class $\nu^* \alpha$. The non-Kähler locus is the union of the ν -exceptional locus and the strict transforms of the curves in the non-Kähler locus of α . By [Bou02, Thm.3.1.24] there exists a modification $\mu' : \tilde{X} \rightarrow X'$ and a Kähler class $\tilde{\alpha}$ on \tilde{X} such that $(\mu')^* \nu^* \alpha = \tilde{\alpha} + E$ where E is an effective \mathbb{R} -divisor. Then $\mu := \nu \circ \mu'$ has the stated properties.

Thus $-E|_{D_l}$ is ample. If $B \subset X$ is an arbitrary curve contained in the image of the exceptional locus, and D_B the support of $\mu^{-1}(B)$, we still have

$$-E|_{D_B} = \tilde{\alpha}|_{D_B} - (\mu^* \alpha)|_{D_B}.$$

Thus we see that $-E$ is μ -ample, in particular its support is equal to the μ -exceptional locus. Since ampleness is an open property we can find an effective \mathbb{Q} -divisor $E' \subset \tilde{X}$ that is μ -ample and such that $-E'|_{D_l}$ is ample for all l . Up to taking a positive multiple we can suppose that E' has integer coefficients. We set

$$\mathcal{K}_m := \mu_* \mathcal{O}_{\tilde{X}}(-mE'),$$

and we claim that there exists a $m \in \mathbb{N}$ such that the restriction $\mathcal{K}_m|_{C_l}$ is ample for all j . Note that this implies that the quotient $(\mathcal{K}_m/\mathcal{K}_m^2)|_{C_l}$ is ample for all j .

Step 2. Proof of the claim. By relative Serre vanishing there exists an $m_0 \in \mathbb{N}$ such that

$$R^i \mu_* \mathcal{O}_{\tilde{X}}(-mE') = 0 \quad \forall i > 0, m \geq m_0$$

and

$$(11) \quad R^i \mu_* \mathcal{O}_{\tilde{X}}(-D_l - mE') = 0 \quad \forall i > 0, m \geq m_0$$

and all l . Moreover we know by [Anc82, Thm.3.1.] that there exists an $m_1 \in \mathbb{N}$ such that the direct image sheaf

$$(\mu|_{D_l})_*(\mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{D_l})$$

is ample for all $m \geq m_1$ and l .

Fix now an $m \geq \max\{m_0, m_1\}$ and consider now the exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-D_l - mE') \rightarrow \mathcal{O}_{\tilde{X}}(-mE') \rightarrow \mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{D_l} \rightarrow 0.$$

Pushing the sequence down to X and using (11) we obtain an exact sequence

$$0 \rightarrow \mu_* \mathcal{O}_{\tilde{X}}(-D_l - mE') \rightarrow \mu_* \mathcal{O}_{\tilde{X}}(-mE') \rightarrow (\mu|_{D_l})_*(\mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{D_l}) \rightarrow 0.$$

Since the inclusion $\mathcal{O}_{\tilde{X}}(-D_l - mE') \rightarrow \mathcal{O}_{\tilde{X}}(-mE')$ vanishes on D_l , its direct image

$$\mu_* \mathcal{O}_{\tilde{X}}(-D_l - mE') \rightarrow \mu_* \mathcal{O}_{\tilde{X}}(-mE')$$

vanishes on $C_l = \mu(D_l)$. In other words we have $\mathcal{O}_{\tilde{X}}(-D_l - mE') \subset \mathcal{J} \cdot \mathcal{O}_{\tilde{X}}(-mE')$ where \mathcal{J} is the full ideal sheaf of C_l . Thus we have an epimorphism

$$\mu_* \mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{C_l} \rightarrow (\mu|_{D_l})_*(\mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{D_l}),$$

which is generically an isomorphism. In particular, $\mu_* \mathcal{O}_{\tilde{X}}(-mE') \otimes \mathcal{O}_{C_l}$ is ample. This proves the claim.

Step 3. Conclusion. The ideal sheaf \mathcal{K}_m defines a 1-dimensional subspace $A \subset X$ such that $C \subset A$. We can now conclude as in the proof of Lemma 7.11: let $\mathcal{I} \subset \mathcal{O}_X$ be the largest ideal sheaf on X that coincides with \mathcal{K}_m in

the generic point of every irreducible curve $C_l \subset C$ such that $[C_l] \in R$. For every curve C_l , the natural map

$$(\mathcal{K}_m/\mathcal{K}_m^2)|_{C_l} \rightarrow (\mathcal{I}/\mathcal{I}^2)|_{C_l}$$

is generically an isomorphism. Since $(\mathcal{K}_m/\mathcal{K}_m^2)|_{C_l}$ is ample and C_l is a curve, this implies that $(\mathcal{I}/\mathcal{I}^2)|_{C_l}$ is ample. Thus $\mathcal{I}/\mathcal{I}^2$ is ample on its support C . By [AT84, Cor.3] there exists a holomorphic map $\nu : \tilde{X} \rightarrow X'$ contracting each connected component of C onto a point. \square

8. RUNNING THE MMP- PROOF OF THEOREMS 1.1 AND 1.2

The following statement is well-known to specialists, we include the proof for the convenience of the reader.

8.1. Proposition. *Let X be a normal \mathbb{Q} -factorial compact Kähler space with at most terminal singularities. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X -negative extremal ray in $\overline{NA}(X)$. Suppose that there exists a morphism $\varphi : X \rightarrow Y$ onto a normal complex space Y such that $-K_X$ is φ -ample and a curve $C \subset X$ is contracted if and only if $[C] \in \mathbb{R}^+[\Gamma_i]$.*

a) *We have an exact sequence*

$$(12) \quad 0 \rightarrow H^2(Y, \mathbb{R}) \xrightarrow{\varphi^*} H^2(X, \mathbb{R}) \xrightarrow{[L] \mapsto L \cdot \Gamma_i} \mathbb{R} \rightarrow 0.$$

In particular we have $b_2(X) = b_2(Y) + 1$.

b) *We have an exact sequence*

$$(13) \quad 0 \rightarrow \text{Pic}(Y) \xrightarrow{\varphi^*} \text{Pic}(X) \xrightarrow{[L] \mapsto L \cdot \Gamma_i} \mathbb{Z}.$$

c) *If the contraction is divisorial, the variety Y has at most terminal \mathbb{Q} -factorial singularities and its Picard number is $\rho(X) - 1$.*

d) *If the contraction is small with flip $X^+ \rightarrow Y$, the variety X^+ has at most terminal \mathbb{Q} -factorial singularities and its Picard number is $\rho(X)$.*

Proof. Note first that the morphism φ is projective since $-K_X$ is φ -ample. Moreover by [KMM87, 1-2-5], [Nak87] we have $R^j \varphi_* \mathcal{O}_X = 0$ for all $j \geq 1$. In particular Y has at most rational singularities.

Proof of statement a). By [KM92, 12.1.3.2] the pull-back $H^2(Y, \mathbb{R}) \rightarrow H^2(X, \mathbb{R})$ is injective. Moreover by [KM92, Thm.12.1.3] the cokernel of the dual map is generated by classes of curves being contracted by φ . Since a curve is contracted by φ if and only if its class is in $\mathbb{R}^+[\Gamma_i]$, the exactness of (12) follows.

Proof of statement b). The non-trivial part of this statement is to prove that if L is a line bundle on X such that $L \cdot \Gamma_i = 0$, there exists a line bundle L' on Y such that $L \simeq \varphi^* L'$. Note that if such a L' exists, we have

$$L' \simeq \varphi_* L.$$

Thus L' is unique up to isomorphism and it is sufficient to prove that the direct image sheaf $\varphi_* L$ is locally free. This is a local property, so fix an

arbitrary point $y \in Y$ and a small Stein neighbourhood $y \in U \subset Y$. Set $X_U := f^{-1}(U)$, then the morphism $f := \varphi|_{X_U} : X_U \rightarrow U$ satisfies the conditions of the relative base point free theorem [Anc87, Thm.3.3]. Thus $L^{\otimes b}|_{X_U}$ is f -globally generated for *all* $b \gg 0$. The base U being Stein we see that $L^{\otimes b}|_{X_U}$ is globally generated for *all* $b \gg 0$. Since L is f -numerically trivial we have

$$L^{\otimes b}|_{X_U} \simeq f^*M, \quad L^{\otimes b+1}|_{X_U} \simeq f^*N$$

for some line bundles M and N on U . In particular we have $f_*(L|_{X_U}) \simeq N \otimes M^*$.

Proof of statements c) and d). These are a well-known consequences of statement b), compare the proof of [Deb01, Prop.7.44]. \square

8.2. Proposition. *In the situation of Proposition 8.1, suppose that X is a threefold. Then we have an exact sequence*

$$(14) \quad 0 \rightarrow N^1(Y) \xrightarrow{\varphi^*} N^1(X) \xrightarrow{[L] \mapsto L \cdot \Gamma_i} \mathbb{R} \rightarrow 0.$$

Before we can prove this statement we need two technical lemmas:

8.3. Lemma. *Let X be a normal compact Kähler space with at most rational singularities. Then there exists a natural (i.e. functorial) injection*

$$i : N^1(X) \hookrightarrow H^2(X, \mathbb{R}).$$

Proof. Note first that if the map $N^1(X) \rightarrow H^2(X, \mathbb{R})$ exists, it is injective. Indeed if $\mu : X' \rightarrow X$ is a resolution of singularities by some compact Kähler manifold, the pull-backs $N^1(X) \rightarrow N^1(X')$ and $H^2(X, \mathbb{R}) \rightarrow H^2(X', \mathbb{R})$ are injective by Lemma 3.3 and the rationality of the singularities, see e.g. [Kir12, Prop.B.2.4]. Moreover the natural map $N^1(X') \hookrightarrow H^2(X', \mathbb{R})$ is injective by classical Hodge theory, so $N^1(X) \hookrightarrow H^2(X, \mathbb{R})$ is injective by functoriality.

Since X is Kähler we can find a base of $N^1(X)$ such that each class is represented by a Kähler form. Thus the map $N^1(X) \rightarrow H^2(X, \mathbb{R})$ exists by the construction in [Gra62, p.346]. \square

8.4. Lemma. *Let X be a normal compact Kähler space with at most rational singularities. Let $\varphi : X \rightarrow Y$ be a morphism with connected fibres onto a normal compact variety Y in class \mathcal{C} which has at most isolated rational singularities. Let $\alpha \in N^1(X) \subset H^2(X, \mathbb{R})$ be a class such that $\alpha = \varphi^*\beta$ with $\beta \in H^2(Y, \mathbb{R})$. Then there exists a smooth real closed $(1,1)$ -form ω_Y on Y such that $\alpha = \varphi^*[\omega_Y]$.*

Proof. Step 1. Reduction to the case where X is smooth and φ a desingularisation. Let $\mu_Y : Y' \rightarrow Y$ be a desingularisation by a compact Kähler manifold. Let $\mu : X' \rightarrow X$ be a desingularisation of X such that we have a factorisation $\varphi' : X' \rightarrow Y'$ satisfying $\varphi \circ \mu = \mu_Y \circ \varphi'$. Then we have $\mu^*\alpha = (\varphi')^* \circ (\mu_Y)^*\beta$ in $H^2(X', \mathbb{R})$. Since X' and Y' are smooth, we have

$$N^1(X') = H^{1,1}(X') \cap H^2(X', \mathbb{R}), \quad N^1(Y') = H^{1,1}(Y') \cap H^2(Y', \mathbb{R}),$$

moreover the pull-back of cohomology classes is a strict morphism in the sense of Hodge theory. Since $\mu^*\alpha \in N^1(X')$, this implies that $(\mu_Y)^*\beta \in N^1(Y')$. Yet Y' is smooth, so there exists a smooth real closed $(1,1)$ -form $\omega_{Y'}$ such that $[\omega_{Y'}] = (\mu_Y)^*\beta$.

Step 2. Suppose that X is smooth and φ a desingularisation. Let ω be a smooth real closed $(1,1)$ -form such that $[\omega] = \alpha$. Fix a point $y \in Y_{\text{sing}}$ and set $E := \varphi^{-1}(y)$. We will prove that there exists a form ω' such that $[\omega'] = \alpha$ and a neighbourhood $E \subset U'$ such that $\omega'|_{U'} = 0$. It is then clear that ω' descends to Y . Let now $U_Y \subset Y$ be a Stein neighbourhood of y that retracts onto y . Note that this implies $H^1(U_Y, \mathcal{O}_{U_Y}) = 0$ and $H^2(U_Y, \mathbb{R}) = 0$. Set $U := \varphi^{-1}(U_Y)$. Since Y has at most rational singularities the Leray spectral sequence implies that $H^1(U, \mathcal{O}_U) = 0$. Consider now the exact sequence

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}_U \rightarrow \mathcal{H}_U \rightarrow 0,$$

where \mathcal{H}_U is the sheaf of pluriharmonic functions on U . By the preceding vanishing we have an injection

$$H^1(U, \mathcal{H}_U) \hookrightarrow H^2(U, \mathbb{R}).$$

Moreover we have by [HL83, Cor.2] an isomorphism

$$H^1(U, \mathcal{H}_U) = H_{\partial\bar{\partial}}^{1,1}(U).$$

In particular we can consider $[\omega|_U]$ as a class in $H^1(U, \mathcal{H}_U)$. Since $\alpha|_U = \varphi^*\beta|_{U_Y}$ with $\beta|_{U_Y} \in H^2(U_Y, \mathbb{R}) = 0$, we see that $[\omega|_U] = 0$. Thus there exists a function $\hat{f} \in \mathcal{A}^0(U)$ such that $\omega|_U = \partial\bar{\partial}\hat{f}$. Let now σ be a bump function around y with support in U_Y . Then we have $(\sigma \circ \varphi) \cdot \hat{f} \in \mathcal{A}^0(X)$ and

$$\partial\bar{\partial}((\sigma \circ \varphi) \cdot \hat{f}) = \omega$$

on some open neighbourhood $E \subset U'$. We set $\omega' := \omega - \partial\bar{\partial}((\varphi^*\sigma) \cdot \hat{f})$. Since this construction is local around y we can repeat it for every y in the finite set Y_{sing} . \square

Proof of Proposition 8.2. By Lemma 3.3 the map $N^1(Y) \xrightarrow{\varphi^*} N^1(X)$ is injective. Suppose now that $L \in N^1(X)$ such that $L \cdot \Gamma_i = 0$. By the exact sequence (12) we have $L = \varphi^*M$ with $M \in H^2(Y, \mathbb{R})$. Note now that Y has at most isolated singularities since the terminal threefold X has at most isolated singularities. Thus we know by Lemma 8.4 that $M \in N^1(Y)$. \square

8.5. Corollary. *Let X be a normal \mathbb{Q} -factorial compact Kähler threefold with at most terminal singularities. Let $\mathbb{R}^+[\Gamma_i]$ be a K_X -negative extremal ray in $\overline{NA}(X)$. Suppose that there exists a bimeromorphic morphism $\varphi : X \rightarrow Y$ such that $-K_X$ is φ -ample and a curve $C \subset X$ is contracted if and only if $[C] \in \mathbb{R}^+[\Gamma_i]$. Then Y is a Kähler space.*

Proof. Let α be a nef supporting class (cf. Proposition 7.3) for the extremal ray $\mathbb{R}^+[\Gamma_i]$. Then $\alpha \cdot \Gamma_i = 0$, so by Corollary 8.2 we have $\alpha = \varphi^*(\alpha')$ with

some $\alpha' \in N^1(Y)$. Since $\alpha \cdot \beta > 0$ for all $\beta \in \overline{NA}(X) \setminus \mathbb{R}^+[\Gamma_i]$ and the map $\varphi_* : \overline{NA}(X) \rightarrow \overline{NA}(Y)$ is surjective by Proposition 3.10, we see that α' is strictly positive on the closed cone $\overline{NA}(Y)$. Hence α' is a Kähler class by Theorem 3.16, once we have verified that Y does not contain an irreducible curve C homologous to 0. This is clear if the extremal ray $\mathbb{R}^+[\Gamma_i]$ is small or φ contracts a divisor to a point. Suppose now that φ contracts a divisor E to a curve $C' \subset Y$; this curve being the only candidate for a curve being homologous to 0. Since the morphism $\varphi|_E : E \rightarrow C'$ is projective, so does E , and thus there exists a curve $C \subset E \subset X$ such that $\varphi(C) = C'$. Thus for some positive number d we have

$$\alpha' \cdot (dC') = \alpha' \cdot \varphi_*(C) = \alpha \cdot C > 0,$$

since $0 \neq [C] \in \overline{NA}(X)$ and $[C] \notin \mathbb{R}^+[\Gamma_i]$. Thus we have $[C'] \neq 0$. \square

Proof of Theorem 1.3. The existence of a morphism $\varphi : X \rightarrow Y$ contracting exactly the curves in the extremal ray is established in the Corollaries 7.7, 7.9 and Theorem 7.14. Moreover Y is a Kähler space by Corollary 8.5. \square

Proof of Theorem 1.1. Step 1: Running the MMP. If K_X is nef, we are finished. If K_X is not nef, there exists by Theorem 1.2 a K_X -negative extremal ray R in $\overline{NA}(X)$. By Theorem 1.3 the contraction $\varphi : X \rightarrow Y$ of R exists. If R is divisorial we can continue the MMP with Y by Proposition 8.1,c). If R is small, we know by Mori's flip theorem [Mor88, Thm.0.4.1] that the flip $\varphi^+ : X^+ \rightarrow Y$ exists, and by Proposition 8.1,d) we can continue the MMP with X^+ .

Step 2: Termination of the MMP. Recall that for a normal compact threefold X with at most terminal singularities, the difficulty $d(X)$ [Sho85] is defined by

$$d(X) := \#\{i \mid a_i < 1\},$$

where $K_Y = \mu^* K_X + \sum a_i E_i$ and $\mu : Y \rightarrow X$ is any resolution of singularities. By [KMM87, Lemma 5.1.16], [Sho85] we have $d(X) > d(X^+)$, if X^+ is the flip of a small contraction. Since the Picard number and the difficulty are non-negative integers, any MMP terminates after finitely many steps. \square

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